Engineering **FIRST YEAR**

Part IA Paper 4: Mathematics

Examples paper 6

(Elementary exercises are marked †, problems of Tripos standard *)

ISSUED ON

27 NOV 2013

Revision question

Evaluate the following integrals:

(a) $\int x^{-1/2} dx$ (b) $\int x^{-1} dx$ (c) $\int \frac{x^5}{1+x^6}\, dx$

(c) $\int x^2 \ln x\, dx$ (d) $\int \sin^4 x \cos x\, dx$ (e) $\int_{-\pi}^{\pi} \sin x \left(x^2 \cos x + x^6\right) dx$

Eigenvalues and Eigenvectors

1 Find the eigenvalues and eigenvectors of the following symmetric matrices:

(i) $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ (v) $\begin{bmatrix} 3 & -4 & 1 \\ -4 & 8 & -4 \\ 1 & -4 & 3 \end{bmatrix}$.

2 If $\underline{y}$ is the reflection of the vector $\underline{x}$ in the plane through the origin with unit normal $\underline{n}$, show that

$$\underline{y} = \underline{x} - (2\underline{x}.\underline{n})\underline{n} ,$$

and hence find the 3×3 matrix R describing the transformation (i.e. find the matrix R such that $\underline{y} = R\underline{x}$). By a geometric argument, what are the eigenvalues and corresponding eigenvectors of R ?

3 Construct the symmetric matrix **A** which has eigenvalues $\lambda_i = 3, 1, {}^1/2$ and corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$. Find also the symmetric matrix B with the *same* eigenvectors, but with the corresponding eigenvalues equal to $1/\lambda_i$. Evaluate the matrix $A\,B$ and comment on your result.

4 A is a real symmetric 2×2 matrix with eigenvalues λ_1 and λ_2 and $f(\lambda) = |A - \lambda I|$ is the polynomial whose roots are the eigenvalues. By considering relevant coefficients in the polynomial $f(\lambda)$ prove that

(a) $A_{11} + A_{22} = \lambda_1 + \lambda_2$ (b) $|A| = \lambda_1 \lambda_2$.

5† Find the eigenvalues and eigenvectors of the 2×2 real, symmetric matrix A where

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}.$$

Hence find a matrix R which changes the coordinate system so that the new axes are aligned with the eigenvectors of A. Calculate the version of the matrix A in these new coordinates, and verify that it agrees with the form derived in lectures.

6 A 3×3 real, symmetric matrix A has eigenvalues λ_i, $i = 1, 2, 3$, and corresponding normalised eigenvectors $\underline{u}_i$ The eigenvalues are real, distinct, and arranged in the order

$$\lambda_1 < \lambda_2 < \lambda_3 .$$

Explain why any vector $\underline{x}$ can be expressed in the form $\underline{x} = \alpha \underline{u}_1 + \beta \underline{u}_2 + \gamma \underline{u}_3$, and hence show that

(a) $\underline{x}^t \underline{x} = \alpha^2 + \beta^2 + \gamma^2$ (b) $\underline{x}^t A \underline{x} = \lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2.$

Use these expressions to show that

$$\lambda_1 \le \frac{\underline{x}^t A \underline{x}}{\underline{x}^t \underline{x}} \le \lambda_3$$

for all vectors $\underline{x}$. When is equality achieved?

7* Using the matrix A of question 6, and again expressing a general vector $\underline{x}$ in terms of the eigenvectors of A, what is $A^n \underline{x}$? What happens to $A^n \underline{x}$ as n gets large? Using the result of question 3, derive a similar result for $(A^{-1})^n \underline{x}$.

With $A = \begin{bmatrix} 3 & -4 & 1 \\ -4 & 8 & -4 \\ 1 & -4 & 3 \end{bmatrix}$ and $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

use Matlab/Octave to calculate $A\underline{x}$, $A^2\underline{x}$, $A^3\underline{x}$ and $A^4\underline{x}$, and hence obtain an approximation for the eigenvalue of A with largest absolute value, and the corresponding eigenvector. Experiment with higher powers of A, and compare your result with the exact answer, which was calculated in question 1 part (v).

Hints

To enter the matrix A into Matlab/Octave, just type `A = [3 -4 1; -4 8 -4; 1 -4 3]`, and similarly for `x`. Matrix/vector arithmetic is then as simple as `A4x = A^4 * x`. To normalize the answer, divide through by `norm(A4x)`. Finally, look at an element-by-element division of `A4x` by `A3x`. With higher powers of A you should get better approximations of the eigenvector and eigenvalue.

Suitable past Tripos questions:
2003 Q4, 2004 Q4, 2005 Q5b (long), 2006 Q2 (short), 2007 Q4 (long), 2008 Q5 (long), 2009 Q3c (long)

Answers

1. (i) eigenvalues 2 & -3 , eigenvectors $[\,2/\sqrt{5}, 1/\sqrt{5}\,]^t$ & $[\,1/\sqrt{5}, -2/\sqrt{5}\,]^t$

   (ii) eigenvalues 4.618 & 2.382 , eigenvectors $[\,0.526, 0.851\,]^t$ & $[\,0.851, -0.526\,]^t$

   (iii) eigenvalues 2 & 3 , eigenvectors $[\,1, 0\,]^t$ & $[\,0, 1\,]^t$

   (iv) eigenvalues $-1, 1$ & -4 , eigenvectors $[\,1, 0, 0]^t$, $[\,0, 1, 0]^t$ & $[\,0, 0, 1]^t$

   (v) eigenvalues $0, 2$ & 12 , eigenvectors $[\,1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\,]^t$, $[\,1/\sqrt{2}, 0, -1/\sqrt{2}\,]^t$
   & $[\,-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\,]^t$

2. $I - 2\underline{n}\,\underline{n}^t$

3. $A = \frac{1}{12}\begin{bmatrix} 23 & 13 & 2 \\ 13 & 23 & -2 \\ 2 & -2 & 8 \end{bmatrix} \qquad B = \frac{1}{6}\begin{bmatrix} 5 & -3 & -2 \\ -3 & 5 & 2 \\ -2 & 2 & 10 \end{bmatrix} \qquad AB = I$

5. Eigenvalues 2 and 6, eigenvectors $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

6. Equality when $\underline{x}$ is a multiple of $\underline{u}_1$ or $\underline{u}_3$

7. $A\underline{x}, A^2\underline{x}, A^3\underline{x}, A^4\underline{x} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 26 \\ -48 \\ 22 \end{bmatrix}, \begin{bmatrix} 292 \\ -576 \\ 284 \end{bmatrix}, \begin{bmatrix} 3464 \\ -6912 \\ 3448 \end{bmatrix}$.

   Approximations for eigenvalue 12.00, eigenvector $[\,0.4092, -0.8165, 0.4073\,]^t$.

Michaelmas 2013

GNW/MPJ