ISSUED ON

15 JAN 2014

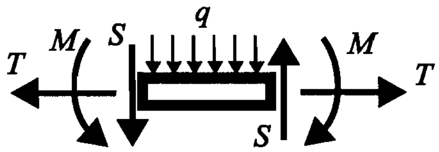
ENGINEERING **FIRST YEAR**

P2: Structural Mechanics, Examples paper 4

Straightforward questions are marked †.
*Tripos standard questions are marked *.*

STATIC EQUILIBRIUM OF BEAMS

Databook sign convention:



Free body analysis of bending moment and shearing force

† 1. The beam shown in Figure 1 is of length L and is simply supported at its ends. It is subjected to a uniformly distributed transverse load of intensity w.

Find the central bending moment, and construct both the bending moment and shearing force plots for the beam.

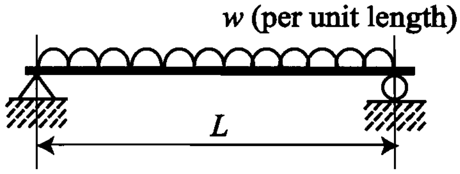


Figure 1

Express the bending moment at the centre of the beam in terms of the *total* load W ($=wL$) and L. Compare this result with that for the case where the entire load is concentrated as a single point force W at the centre of the same beam. Why is the bending moment greater for the concentrated load?

2. The simply-supported beam of length L that is shown in Figure 2 carries a transverse force W at distance a from one support. By considering the equilibrium of a suitably cut piece of the beam, obtain an expression for the bending moment in the beam *at the point of application of the force* in terms of W, L and a. Make a plot of this bending moment, for given W, L, as a varies in the range $0 < a < L$. Which load-position gives the largest bending moment in the beam?

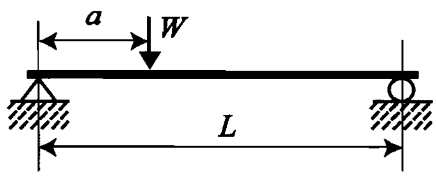


Figure 2

3 The structure shown in Figure 3 consists of two beams, AB and BCD, connected by a frictionless pin at B. Determine the bending moments at A and C, and hence draw the bending moment diagram.

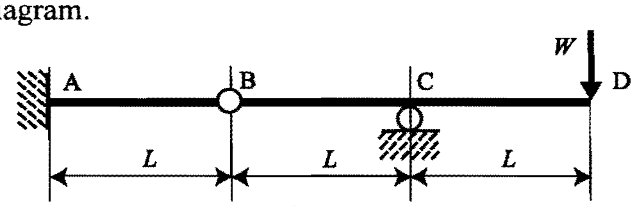


Figure 3

* 4. The simply-supported beam in Figure 4(a) is loaded by a couple C at a particular place, as shown. One way of applying the couple is to have equal horizontal forces applied to the ends of short vertical stubs welded onto the beam, as shown.

Construct plots of M and S: work separately from each end towards the middle.
[The support arrangements are drawn more elaborately than in previous questions to emphasize that the supports may now have to provide (vertical) reactions either up or down.]

Figures 4(b), (c) and (d) show three different loadings of the same beam, all involving pure couples. Construct plots of M and S for each loading case.

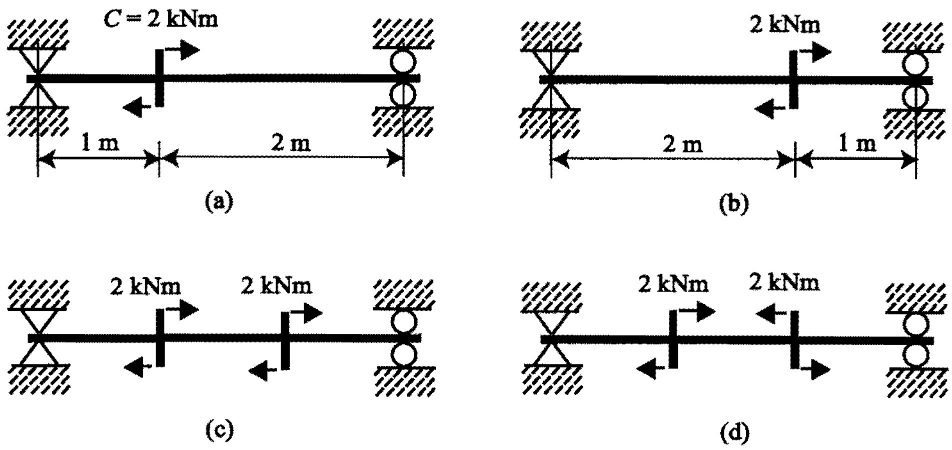


Figure 4

Differential equations of equilibrium

† 5. Figure 5 shows the distribution of bending moment M in a beam which is simply supported at $x = 0$ and $x = 4$. In the portions $0 < x < 2$ and $2 < x < 4$, respectively, $M(x)$ is given by the two formulas that are displayed. The beam extends a short distance beyond the region $0 < x < 4$ at either end, with $M = 0$ in each, as indicated.

Use the differential equilibrium equations in order to find $S(x)$ and then $q(x)$. How does a *discontinuity* (jump) in $S(x)$ come about? Make a sketch to show the forces that act upon the beam, in order to produce this distribution of bending moment.

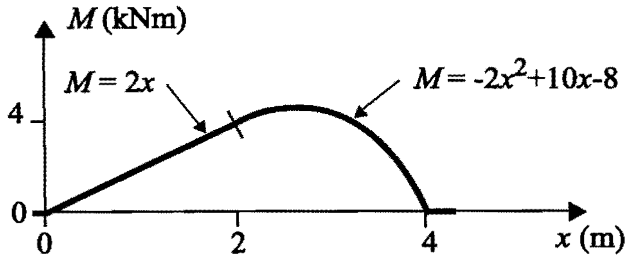


Figure 5

6. Figure 6 shows a simply-supported beam of length L, and an x-coordinate system. The non-uniform loading has intensity q:

$$q = q_0 \sin (2\,\pi x/L).$$

By integrating differential equilibrium equations, and considering boundary conditions, obtain expressions for $S(x)$ and $M(x)$. Sketch both of these functions.

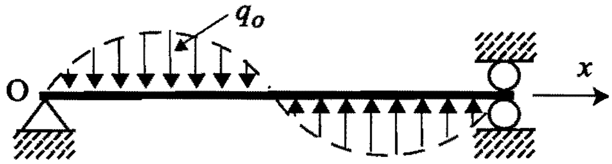


Figure 6

7. A beam of length 3 m is simply supported at its ends and is loaded by a linearly varying distributed load, whose intensity varies from 0 kN/m at x = 0 to 2 kN/m at x = 3m, as shown in Figure 7. [Actually, the beam is vertical. It forms part of a wall that retains water, and is supported laterally at the base and at the top. But here we think of it as horizontal, for the purpose of finding bending-moments, etc.]

(a) Find the total, resultant force exerted by the loading q. Where does it act? [Hint: either perform the appropriate integrations or use standard geometrical results concerning triangles.]

(b) Determine the two support reactions, and hence the shearing force S at $x = 0$.

(c) Solve the equilibrium equation $dS/dx = q$, using the boundary condition (b) to fix the constant of integration. Sketch $S(x)$.

(d) Solve the equilibrium equation $dM/dx = S$, using the condition at $x = 0$ to fix the constant of integration.

(e) Find the value of x where M is maximum/minimum, and hence find the value of M there.

(f) Make a sketch of M(x).

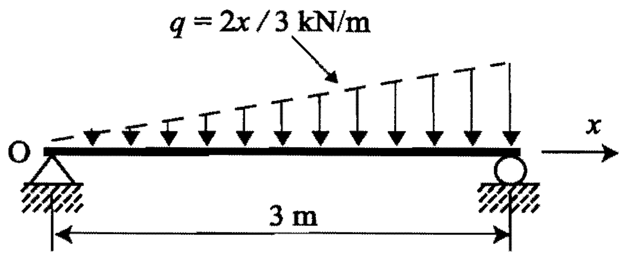


Figure 7

Bending - moment diagrams for simple frames

† 8. Figure 8 shows a plane structure which is subjected in turn to four different loading conditions. The structure consists of a vertical post that is rigidly built into the ground, and a horizontal bar which is rigidly connected to the top of the post.

By considering the equilibrium of suitable free bodies, determine the bending moment throughout the structure for each of the four loading cases. In each case draw the bending-moment diagrams on the picture of the structure, using the convention that M is plotted on the "tensile side" of the member.

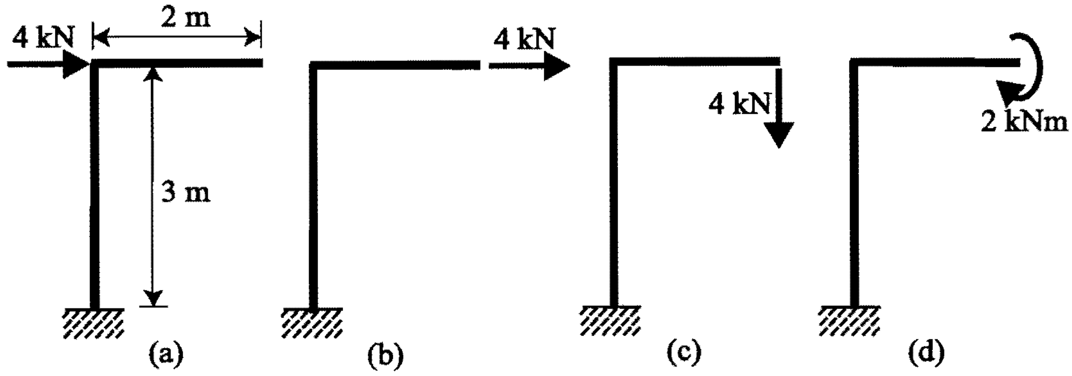


Figure 8

* 9. The plane frame shown in Figure 9 consists of three bars connected by frictionless pins at B, C and D. At A the vertical bar is built into the rigid base

(a) Draw separate free-body diagrams for the three separate pieces, and use equilibrium equations to find the horizontal and vertical components of the reactions at all pins.

(b) Find the bending moments along the three members and draw the bending-moment diagram on the original structure, using the same convention as in Q. 8.

(c) Repeat for the same structure, but now sustaining a horizontal force of 6 kN acting to the right on member AB at a height of 2 m above the base, in addition to all the loads shown in Figure 9.

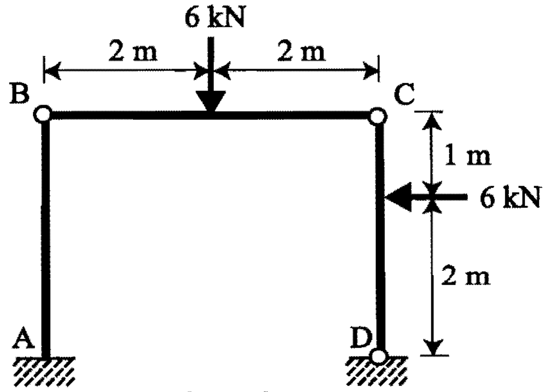


Figure 9

DEFLECTION OF STRAIGHT ELASTIC BEAMS

Curvature-rotation relationship

† 10. A bi-metallic strip, 0.5 m long, is *straight* when its temperature is 0° C. When a portion of the strip is at $\theta\ ^{\circ}$ C it has curvature κ proportional to θ and given by

$$\kappa = 0.005\ \ \theta \text{ rad m}^{-1}$$

Curvature has a positive sense when the centre of curvature lies *below* the beam. The strip is mounted at $0\ ^{\circ}$ C as a horizontal cantilever, with the left-hand end clamped to a rigid wall and the other end free, as shown in Figure 10(a). The weight of the strip is negligible.

Determine the *rotation* of the free end of the cantilever when the strip is subjected to each of the three temperature distributions shown in Figure 10(b-d).

By making sketches of the three deflected shapes (but without doing any calculations) say which of the three cases you expect to have the largest vertical *deflection* at the tip.

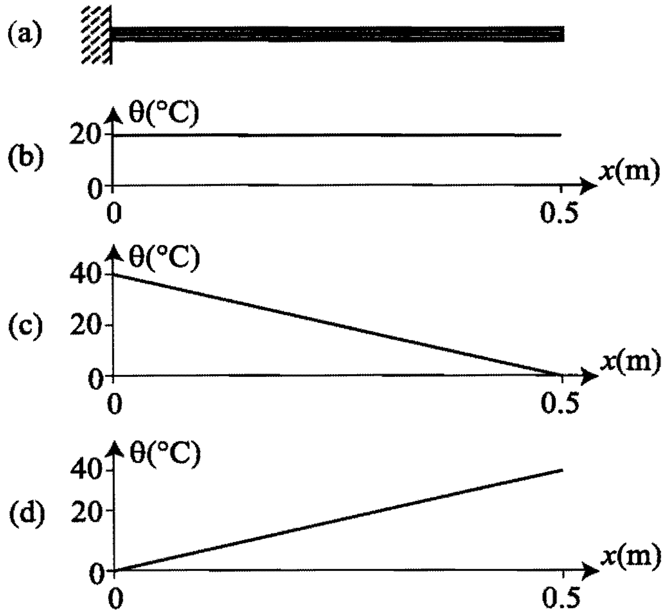


Figure 10

Differential equations for deflection of beams

† 11. (a) The elastic cantilever beam shown in Figure 11(a) has uniform bending stiffness B [Nm^2] and is straight when unloaded. It carries a concentrated transverse load W [N] at its tip, as shown.

By considering the equilibrium of a free body resulting from a cut made at distance x from the left-hand end, obtain an expression for $M(x)$ [Nm]. Then use the elastic bending law $M = B\kappa$ to obtain an expression for $\kappa(x)$ [m^{-1}]. Lastly, use the geometric relation

$$\kappa = -d^2v/dx^2$$

(where v is the transverse displacement) and integrate, using appropriate boundary conditions to obtain formulas for: (i) the deflection of the tip, $x = L$, due to W, and (ii) the rotation of the beam at its tip.

(b) Repeat the calculation for the situation shown in Figure 11(b), where now the (total) load W is distributed uniformly along the beam.

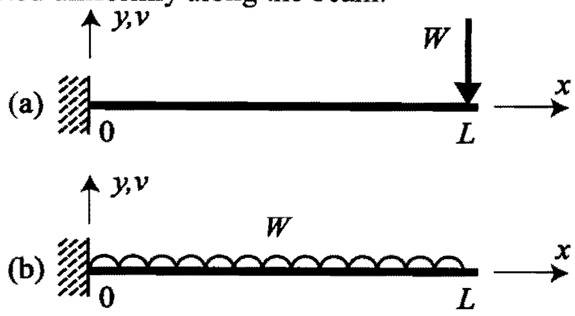


Figure 11

† 12. The uniform elastic cantilever shown in Figure 12(a) has total length $a + b$. It is straight when unloaded. Its flexural stiffness is B.

(a) By considering the beam as a cantilever of length a, to which is attached a straight "pointer" of length b (straight because $M = 0$ in this portion), and using the results of Q.11, find an expression for the vertical displacement of the tip.

(b) Repeat for the same cantilever, but now with the loading shown in Figure 12(b).

(c) Figure 12(c) shows a variant of (a) in which there is a welded right-angle bend. In this case obtain expressions for both the vertical and the horizontal components of displacement of the tip. (Note that the deflections are all small in comparison with the length of the beam).

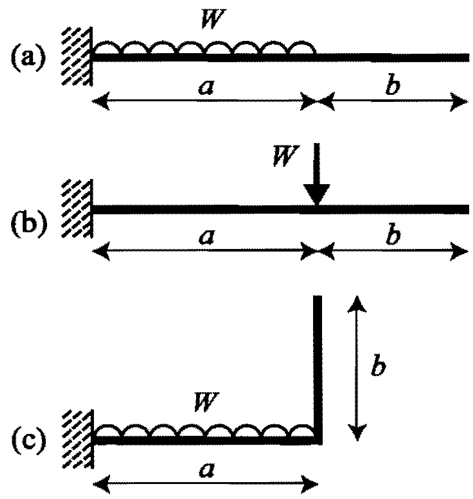


Figure 12

13. The uniform elastic beam shown in Figure 13(a) is of length L [m] and its bending stiffness is B [Nm^2]. It is simply supported at its ends, and is straight when not loaded. It is now subjected to a non-uniform distributed transverse loading of intensity

$$q = q_0 \sin(\pi x/L) \quad [\mathrm{Nm}^{-1}].$$

(a) Using the x-coordinate shown in the diagram, and the differential-equation method, obtain an expression for the bending moment $M(x)$ [Nm] in the beam.

(b) Using the elastic law, obtain an expression for the distribution of curvature $\kappa(x)$ [m^{-1}] in the beam.

(c) Putting $\kappa = -d^2v/dx^2$, integrate twice with respect to x to obtain an expression for the vertical component of deflection $v(x)$ in the beam; use two suitable boundary conditions to fix the two constants of integration.

(d) Hence obtain a formula for the transverse deflection at the centre of the beam due to the given loading.

(e) Show that for a loading $q = q_0 \sin(n\pi x/L)$, as in Figure 13(b), the transverse deflection is given by $v = -(q_0L^4/n^4B\pi^4)\sin(n\pi x/L)$.

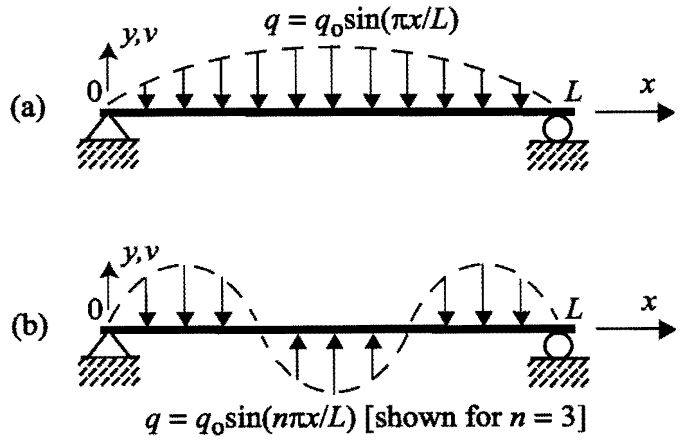


Figure 13

Static Equilibrium of Beams: Suitable Questions from IA Tripos Papers (Paper 2)

2006	Q4
2007	Q3
2009	Q3
2010	Q2
2011	Q1
2012	Q1, Q4
2013	Q3

ANSWERS

1. $-wL^2/8$; $S = -wL/2 + wx$; $M = -wx(L-x)/2$; $-WL/8$ vs. $-WL/4$.

2. $M_{\text{under}W} = -\ Wa(L-a)/L$; $a = L/2$.

3. $M_A = -WL$; $M_C = +WL$.

4. (a) $S = 2/3$; $M = 2x/3$ $(0 < x < 1)$, $= 2x/3 - 2$ $(1 < x < 3)$.
   (b) $M = 2x/3$ $(0 < x < 2)$, $= 2x/3 - 2$ $(2 < x < 3)$.
   (c) $M = 4x/3$ $(0 < x < 1)$; $= 4x/3 - 2$ $(1 < x < 2)$; $= 4x/3 - 4$ $(2 < x < 3)$.
   (d) $M = 0, -2, 0$ in the three parts. [Units: kN, kNm]

5. Reactions 2, 6 kN (both down) at ends ; $q_o = -4$ kN/m (i.e. up) on right-hand part.

6. $S = -(q_o L/2\pi) \cos 2\pi x/L$; $M = -q_o\,(L/2\pi)^2 \sin 2\pi x/L$.

7. (a) 3 kN, at $x = 2$ m; (b) -1 kN; (c) $S = x^2/3 - 1$ kN; (d) $M = x^3/9 - x$ kNm;
   (e) $M_{min} = -1.15$ kNm at $x = 1.73$ m.

8. (a), (b) M = zero in horizontal member, varies linearly in vertical member ;
   (c) M varies linearly along horizontal member; uniform in vertical member ;
   (d) M uniform throughout.

9. $M = 0$ at B, C, D.
   (b) $|M| = 6$ kNm under vertical force, $= 4$ under horizontal force, $= 12$ at A.
   (c) $|M| = 6$ kNm under vertical force, $= 4$ under horizontal forces, $= 0$ at A .

10. 0.05 rad = 2.87^{o}; (c).

11. (a) $v = -WL^3/3B$; $dv/dx = -WL^2/2B$.
    (b) $v = -WL^3/8B$; $dv/dx = -WL^2/6B$.

12. (a) $Wa^2(a/8 + b/6)/B$ down; (b) $Wa^2(a/3 + b/2)/B$ down;
    (c) $Wa^3/8B$ down, $Wa^2b/6B$ to the right.

13. (a) $M = -(\,q_o\, L^2/\pi^2\,) \sin \pi x/L$, (b) $\kappa = M/B = ...$
    (c) $v = -(\,q_o\, L^4/B\pi^4\,) \sin \pi x/L$, (d) $v_{centre} = -(q_o\, L^4/B\pi^4\,)$.

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January 2014

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