

Part IA Paper 4: Mathematical Methods

Examples Paper 7

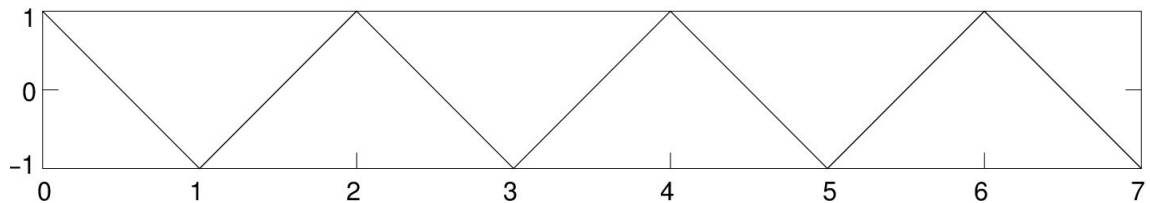
*Elementary exercises are marked †, problems of Tripos standard *.
Hints and answers can be found at the back of the paper.*

Steps and impulses

1. † What familiar quantities do the following functions represent?
 - (i) A time-varying force $f(t) = F\delta(t)$.
 - (ii) A line distribution of loading $w(x) = P\delta(x - a)$ per unit length.

2. † Explain briefly the meaning of $\delta(t)$ and $\delta(t - a)$. If $f(t)$ is a continuous function, explain why $\int \delta(t - a)f(t) dt = f(a)$, provided $t = a$ lies in the range of integration. Evaluate
 - (i) $\int_0^5 \delta(t - 3) dt$
 - (ii) $\int_{-1}^1 \delta(x) \sin x dx$
 - (iii) $\int_0^1 \delta(x - \pi/4) \exp[\cos(1/x)] dx$
 - (iv) $\int_0^1 \delta(x + \pi/4) \cos x dx$

3. † Calculate and sketch the first and second derivatives of the following functions, paying particular attention to points of discontinuity:
 - (i) the periodic triangular wave, a portion of which is plotted below:



(ii) $f(t) = \begin{cases} 0 & t < 0 \\ \sin \Omega t & t \geq 0 \end{cases}$

4. Consider the following two differential equations:

(i) $\frac{dy}{dt} + 3y = f(t)$

(ii) $\frac{d^2y}{dt^2} + 4y = f(t)$

The input is a unit step function, i.e. $f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$, and the output $y(t)$ is zero for $t < 0$.

Explain why $y(t)$ is continuous at $t = 0$ in (i), and both $y(t)$ and $\dot{y}(t)$ are continuous at $t = 0$ in (ii). Calculate the solution $y(t)$ in each case. Hence calculate the solution of each equation with input $f(t) = \delta(t)$ when the output $y(t)$ is zero for $t < 0$.

5. * For a certain system, the output $y(t)$ to an input $f(t)$ satisfies the differential equation

$$\frac{dy}{dt} + \alpha y = f(t)$$

- (i) Find the step response of the system, by solving the equation with $f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ and $y = 0$ for $t \leq 0$.
- (ii) Hence write down the output of the system to input $f(t) = \begin{cases} 0 & t < 0 \\ 1/T & 0 \leq t \leq T \\ 0 & t > T \end{cases}$, with $y = 0$ for $t \leq 0$. Verify that this becomes the impulse response in the limit $T \rightarrow 0$.

Convolution

6. For the system whose input-output characteristics are described by the equation studied in question 4(i), find by convolution the output $y(t)$ in response to the input

(i) $f(t) = \begin{cases} 0 & t < 0 \\ e^{-2t} & t \geq 0 \end{cases}$

(ii) $f(t) = \begin{cases} 0 & t < 0 \\ \sin t & t \geq 0 \end{cases}$

Assume that the output of the system is zero for $t < 0$. In both cases, verify by substitution that the result you obtain really does satisfy the differential equation of question 4(i).

7. * For a certain linear system, the output $y(t)$ to an input $f(t)$ satisfies the differential equation

$$\frac{1}{12} \left(2 \frac{d^2 y}{dt^2} + 11 \frac{dy}{dt} + 12y \right) = f(t)$$

- (i) Find the step response of the system, by solving the equation with $f(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ and $y = \frac{dy}{dt} = 0$ for $t = 0$.
- (ii) Find the impulse response of the system.
- (iii) Hence find by convolution the system response to input $f(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$ assuming $y = \frac{dy}{dt} = 0$ for $t = 0$.
- (iv) Suppose the differential equation describes a car's suspension, with the input $f(t)$ representing the height of the road and the output $y(t)$ representing the height of the chassis. The car is to be driven over a rough track with height profile given in the file `f.dat`¹. The file contains 10000 numbers. Each number is the height of the track at 50 ms intervals, as the car drives over it at constant speed. Use Matlab/Octave to plot (a) the road height as a function of time, (b) the suspension's step response, (c) the suspension's impulse response and (d) the height of the chassis as the car drives over the track. You might find it illuminating to plot $f(t)$ and $y(t)$ on the same axes, zooming in on, say, the first 25 s.

¹Available from <http://www-sigproc.eng.cam.ac.uk/~jl>

8. * An elastic band with tension T is fixed at points $x = 0, L$. A transverse force $f(x)$ per unit length is applied, and small transverse displacements of the band $y(x)$ are produced.

- (i) A point force $f(x) = \delta(x - a)$ is applied at $x = a$ on the band. Sketch the form of the displacement of the band. Using a force-balance argument (and assuming that the displacement is small), show that the displacement is approximately

$$g(x, a) = \begin{cases} \frac{(L - a)x}{LT} & x \leq a \\ \frac{a(L - x)}{LT} & x \geq a \end{cases}$$

- (ii) Explain briefly why the displacement in response to a general applied force is

$$y(x) = \int_0^L g(x, a)f(a) da$$

- (iii) Hence find the displacement of the band when a uniform force is applied over half of the length, so that $f(x) = \begin{cases} F & 0 \leq x \leq L/2 \\ 0 & L/2 \leq x \leq L \end{cases}$

The material on this examples paper has been included in the mathematics course since 1993, and further questions can be found in Part IA Paper 4 in each year since then.

Timing of lectures

Questions 1–3 can be done when Section 1 of handouts has been completed (late January 2014)

Questions 4–7 can be done when Sections 2 and 3 of handouts have been completed (early February 2014)

Question 8 can be done when Section 4 of handouts has been completed (mid February 2014).

Hints

7. (iv) (a) Use the `load` command to read in the data and the `plot` command to plot it. You can examine the data values by simply typing the name of the variable, `f` in this case.

(iv) (b) We have already found the step response algebraically, but Matlab/Octave deals primarily with numerical data, not symbolic algebra. So we need to sample the step response at 50 ms intervals, to match the 50 ms sampling of the road height profile in `f`. We can generate the sample time points using the notation `t = [a:b:c]`, which produces a vector starting at `a`, finishing at `c`, at intervals of `b`. In this case, `a` should be 0, `b` should be 0.05 and `c` should be large enough to capture the step response up to the point where it has levelled off: experiment with different values of `c`. We can then evaluate arbitrary functions at these time points. For example, `h = sin(t)` evaluates the function $\sin t$ at the time points in `t`: try it and plot the resulting `h`. Use this approach to calculate and plot the step response.

- (iv) (c) Use the `t` vector from (b) to evaluate and plot the impulse response `g`.

(iv) (d) The Matlab/Octave command `conv(f, g)` performs numerical convolution by multiplying and summing the elements of `f` and `g`. It does not take into account the time interval between the samples, so for a correctly scaled result we would need to calculate `0.05*conv(f, g)`. To focus on the first 25 s of data, we must look at `y(1:500)` and `f(1:500)`.

Answers

- (i) An impulse of magnitude F at $t = 0$
 (ii) A point load of magnitude P at $x = a$
- (i) 1 (ii) 0 (iii) $\exp[\cos(4/\pi)] = 1.3407$ (iv) 0
- (i) $\frac{df}{dt}$ is a square wave, with values $-2, 0 \leq t < 1$ and $2, 1 \leq t < 2$, then repeating periodically.

$$\frac{d^2f}{dt^2} = \dots + 4\delta(t+1) - 4\delta(t) + 4\delta(t-1) - 4\delta(t-2) \dots$$
 (ii) $\frac{df}{dt} = \begin{cases} 0 & t < 0 \\ \Omega \cos \Omega t & t > 0 \end{cases}$ $\frac{d^2f}{dt^2} = \Omega\delta(t) + \begin{cases} 0 & t < 0 \\ -\Omega^2 \sin \Omega t & t \geq 0 \end{cases}$
- (i) Step response $y = \begin{cases} 0 & t \leq 0 \\ \frac{1 - e^{-3t}}{3} & t \geq 0 \end{cases}$; impulse response $y = \begin{cases} 0 & t < 0 \\ e^{-3t} & t \geq 0 \end{cases}$
 (ii) Step response $y = \begin{cases} 0 & t \leq 0 \\ \frac{1 - \cos 2t}{4} & t \geq 0 \end{cases}$; impulse response $y = \begin{cases} 0 & t < 0 \\ \frac{\sin 2t}{2} & t \geq 0 \end{cases}$
- (i) $y = \begin{cases} 0 & t \leq 0 \\ \frac{1 - e^{-\alpha t}}{\alpha} & t \geq 0 \end{cases}$; (ii) $y = \begin{cases} 0 & t < 0 \\ \frac{1 - e^{-\alpha t}}{\alpha T} & 0 \leq t \leq T \\ \frac{(e^{\alpha T} - 1)e^{-\alpha t}}{\alpha T} & t > T \end{cases}$
- (i) $e^{-2t} - e^{-3t}$; (ii) $(3 \sin t - \cos t + e^{-3t})/10$
- (i) $y = \frac{3}{5}e^{-4t} - \frac{8}{5}e^{-3t/2} + 1$
 (ii) $y = \frac{12}{5}(e^{-3t/2} - e^{-4t})$
 (iii) $\frac{4}{5}(e^{-4t} - 6e^{-3t/2} + 5e^{-t})$
- (iii) $y = \begin{cases} \frac{Fx(3L - 4x)}{8T} & 0 \leq x \leq L/2 \\ \frac{FL(L - x)}{8T} & L/2 \leq x \leq L \end{cases}$