

EGT0
ENGINEERING TRIPOS PART IA

Tuesday 12 June 2018 9:00 to 12:10

Paper 4

MATHEMATICAL METHODS

Answer *all* questions.

The **approximate** number of marks allocated to each part of a question is indicated in the right margin.

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number **not** your name on the cover sheet.

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM

CUED approved calculator allowed

Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

1 **(short)** Find the solutions of $z^{2i} + 2 = 3z^i$ in Cartesian form. [10]

2 **(short)** Find $\lim_{y \rightarrow \infty} y \log(1 + \frac{x}{y})$. [10]

3 **(short)** Find all solutions of the difference equation $x_{n+2} + x_{n+1} - 6x_n = 0$, subject to $x_0 = x_1$. [10]

4 **(long)** In three-dimensional space \mathbf{x} satisfies

$$4\mathbf{x} \cdot \mathbf{m} = \mathbf{x} \cdot \mathbf{x} - (\mathbf{x} \cdot \mathbf{m})^2 - (\mathbf{x} \cdot \mathbf{n})^2, \quad (1)$$

where \mathbf{m} and \mathbf{n} are two orthogonal unit vectors.

(a) Explain why it must be possible to choose α, β, γ such that

$$\mathbf{x} = \alpha\mathbf{m} + \beta\mathbf{n} + \gamma\mathbf{m} \times \mathbf{n}. \quad [5]$$

(b) Show that if \mathbf{x} satisfies eq. (1) then

$$4\alpha = \gamma^2$$

and give the range of values for β . Provide a geometrical interpretation for eq. (1). [10]

(c) For any \mathbf{x} satisfying eq. (1) obtain expressions for the shortest distance between:

(i) \mathbf{x} and the plane $\mathbf{r} \cdot \mathbf{m} = -1$;

(ii) \mathbf{x} and the line $\mathbf{r} = \mathbf{m} + \lambda\mathbf{n}$.

How are these distances related? [15]

5 (long) Let

$$A = \begin{bmatrix} 1 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & 3 \end{bmatrix}.$$

- (a) Find the eigenvalues of A . [10]
- (b) Find a normalized (unit length) eigenvector for the median eigenvalue of A . [10]
- (c) Find the eigenvalues of A^{-1} . [5]
- (d) Find $\det(A^3)$. [5]

SECTION B

6 (**short**) Consider the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 1$$

with initial conditions $x(0) = \dot{x}(0) = 0$. Show that the Laplace transform of $x(t)$ can be written in the form

$$X(s) = \frac{A}{s} + \frac{B}{[(s+1)^2 + 4]} + \frac{C(s+1)}{[(s+1)^2 + 4]}$$

where A , B and C are constants. Hence find $x(t)$. [10]

7 (**short**) A linear system has impulse response $e^{-t} - e^{-2t}$ for $t \geq 0$. Find its response to the input

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$

Is this a first or second order system? Justify your answer. [10]

8 (**short**) A tennis match is contested between two players, A and B. The match comprises a number of *sets*. The winner of the match is the first player to win three sets. The probability that player A wins any one set is a .

Find an expression for the probability P that player A wins the match. Verify that the value of P is as you would expect for three distinct values of a . [10]

9 (long) The function $y(x)$ in Fig. 1 is odd and has period $2L$.

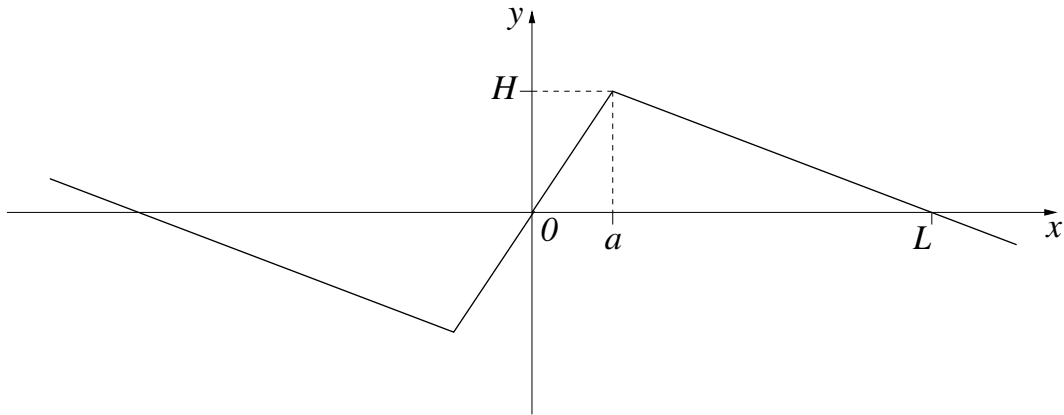


Fig. 1

(a) Show that

$$y(x) = \sum_{n=1}^{\infty} \frac{2HL^2}{n^2\pi^2a(L-a)} \sin\left(\frac{n\pi a}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

is a Fourier series representation of $y(x)$. [15]

(b) Find a Fourier series for $y(x)$ in the limit as $a \rightarrow 0$. You should either take the limit of the expression in (a), or manipulate the Fourier series for the saw-tooth wave in the *Mathematics Data Book*. [10]

(c) Comment on the rates of convergence of the series in (a) and (b). [5]

10 (**long**) The temperature in a room is given by

$$T = 20 + A \sin [(xy + x + y + 1 - z)/d]$$

where T is the temperature in degrees Celsius, (x, y, z) are the Cartesian coordinates in metres, the amplitude $A = 1$ °C and the scale factor $d = 1$ m.

- (a) Find an expression for ∇T . What are the units of ∇T ? [4]
- (b) Determine the equation of the plane that is tangent to the constant-temperature surface at the point $(1, 0, 2)$. [6]
- (c) Sketch contours of constant temperature in the plane $z = 0$. [6]
- (d) A bee is flying along a spiral trajectory. Its position is given by $B(\cos 2t, \sin 2t, 2t/\pi)$ metres, where t is the time in seconds and the amplitude $B = 1$ m. Find the rate of change of temperature experienced by the bee as it passes through the point $(1, 0, 2)$. [8]
- (e) A wasp flies at constant speed 1 ms^{-1} through the point $(1, 0, 4)$, along a straight-line trajectory in the direction $[1 \ 0 \ 1]^t$. Sketch a graph showing the temperature experienced by the wasp as a function of time. [6]

SECTION C

11 (**short**) The titles of ten children's books and their average reader ratings are stored in the Python data structure:

```
Kidsbooks = {"Peppa Pig": 4.4, "Peter Rabbit": 4.2, "Gruffalo": 4.9,
             "Gingerbread Man": 1.1, "Goldilocks": 4.1, "Bedtime Stories":
             3.5, "Snow White": 3.9, "My Little Dog": 4.3, "Animals": 2.0,
             "Three Little Pigs": 3.7}
```

(a) Describe which data structure is used above, and give an example of a task that can be completed efficiently using this data structure. [2]

(b) An inexperienced developer has written the following fragment of Python to implement a feature. Describe the purpose of this code fragment, and give the output of the print statement.

```
v = list(Kidsbooks.values())
k = list(Kidsbooks.keys())
r = 3
result = []
for i in range(r):
    max_v = max(v)
    # return the index in list v at which max_v appears
    index_max_v = v.index(max_v)
    result.append(k[index_max_v])
    k.remove(k[index_max_v])
    v.remove(max_v)
print(result) [6]
```

(c) The parameter `r` is now set to `len(Kidsbooks)`. Describe what the new feature does. [2]

12 (**short**) A class definition in Python of a BankAccount is provided below.

```
class BankAccount:
    def __init__(self, balance, account_holder):
        self.balance = balance
        self.account_holder = account_holder
    def credit(self, amount):
        self.balance = self.balance + amount
    def __repr__(self):
        return str(self.account_holder) + ": " + str(self.balance)
```

(a) Identify class attributes and methods, and describe the class functionality. [2]

(b) Extend the class with a method called debit to reduce the balance by a given amount, provided it does not exceed the balance. Print "Not sufficient funds" if the balance is less than the amount to be debited. [5]

(c) Give the output produced by the following code which uses the BankAccount class.

```
newAccount = BankAccount (900, "James Smith")
print(newAccount)
```

```
newAccount.credit(50)
print(newAccount.balance)
```

```
newAccount.debit(1000)
print(newAccount) [3]
```

END OF PAPER