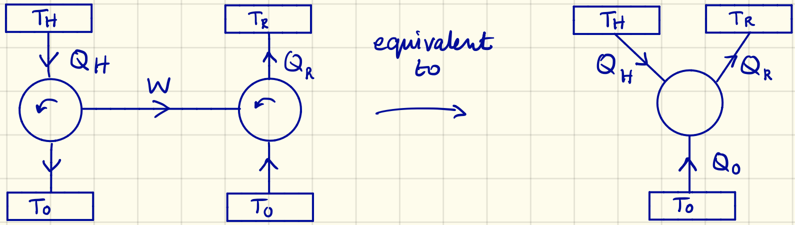
1.

(a)



1st Law: $Q_H + Q_0 = Q_R$

2nd Law: $\dfrac{Q_H}{T_H} + \dfrac{Q_0}{T_0} = \dfrac{Q_R}{T_R}$

Eliminate Q_a:

$$\frac{Q_H}{T_H} + \frac{Q_R}{T_0} - \frac{Q_H}{T_0} = \frac{Q_R}{T_R}$$

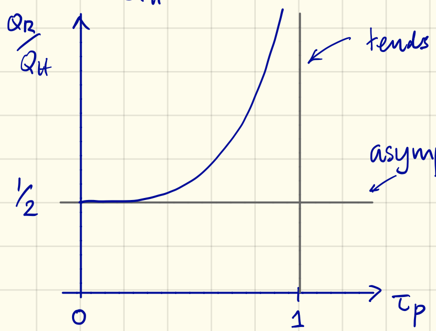
$$\Rightarrow \frac{T_0}{T_H} + \frac{Q_R}{Q_H} - 1 = \frac{T_0}{T_R}\frac{Q_R}{Q_H}$$

$$\Rightarrow \frac{Q_R}{Q_H}\left(1 - \frac{T_0}{T_R}\right) = 1 - \frac{T_0}{T_H}$$

This question was well answered by most students

$$\Rightarrow \frac{Q_R}{Q_H} = \frac{1-\tau_E}{1-\tau_p} \quad \text{where} \quad \tau_E = T_0/T_H \quad \text{and} \quad \tau_p = T_0/T_R$$

(b) $Q_R/Q_H = \frac{1}{2}(1-\tau_p)^{-1}$; $\tau_p = T_0/T_R$ where $\infty \geq T_R \geq T_0 \Rightarrow \tau_p \in [0,1]$



tends to ∞ as $\tau_p \to 1 \Rightarrow T_R \to T_0 \Rightarrow$ small temp. rise

asymptotes to 1/2 when $\tau_p \to 0$ when T_R is v. big

Few students answered this thoughtfully. The user cannot alter τ_E.

(c) For max. heat input to the room, Q_R/Q_H, one wants $\tau_E \to 0$ (high temp. diff across engine) and $\tau_p \to 1$ (low temp diff across pump). τ_E is fixed by power station. τ_p fixed by user. $\therefore$ aim for small $T_R - T_0$.

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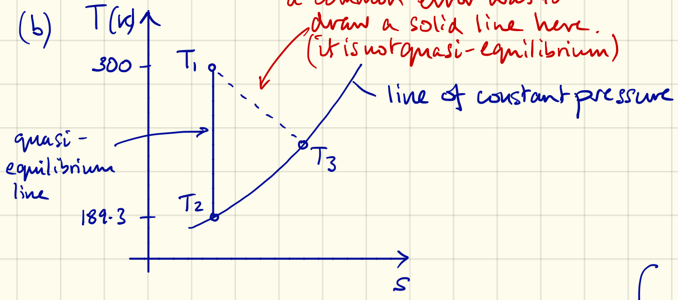
2.

(a) Isentropic perfect gas $\Rightarrow$ $T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 300\left(\frac{1}{10}\right)^{1/5} = 189.3$ kelvin

(b)



Most students answered this question well.

(c) $s_3 - s_1 = c_p \ln\left(\frac{T_3}{T_1}\right) - R\ln\left(\frac{p_3}{p_1}\right)$

$$\left\{ \begin{array}{l} \frac{T_3}{T_1} = \frac{246}{300} \; ; \; \frac{p_3}{p_1} = \frac{1}{10} \\ R = c_p - c_v = c_p\left(\frac{\gamma-1}{\gamma}\right) \end{array} \right.$$

$$= 1000\left\{\ln\left(\frac{246}{300}\right) - \frac{0.25}{1.25}\ln\left(\frac{1}{10}\right)\right\}$$

$$= 262.1 \text{ J kg}^{-1}\text{ K}^{-1}$$

Either consider ideal gas law:

$$\frac{p_1 V_1}{T_1} = \frac{p_3 V_3}{T_3} \Rightarrow \frac{V_3}{V_1} = \frac{p_1 T_3}{p_3 T_1} = 10 \times \frac{246}{300} = 8.2$$

Or consider the gas inside the extinguisher as a system that expands against atmospheric pressure, adiabatically but not reversibly, from volume V_1 to volume V_3. 1st law gives:

$$-p_3(V_3 - V_1) = c_v(T_3 - T_1)$$

$$\Rightarrow -\frac{p_3 V_1}{c_v T_1}\left(\frac{V_3}{V_1} - 1\right) = \frac{T_3}{T_1} - 1$$

$$\Rightarrow -\frac{p_3}{p_1}\frac{R}{c_v}\left(\frac{V_3}{V_1} - 1\right) = \frac{T_3}{T_1} - 1$$

$$\Rightarrow \frac{V_3}{V_1} = 1 - \frac{p_1}{p_3}\frac{c_v}{R}\left(\frac{T_3}{T_1} - 1\right) = 1 - \frac{10}{\gamma-1}\left(\frac{246}{300} - 1\right) = 8.2$$

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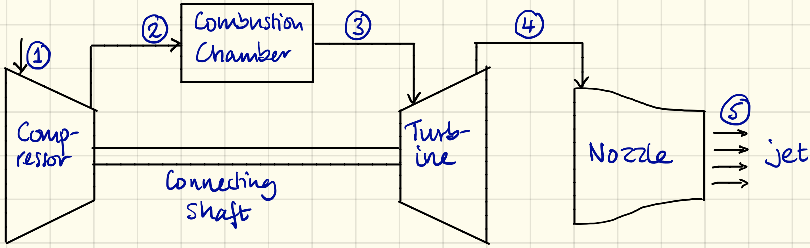
3 ①



$$r_p = p_2/p_1 \quad ; \quad \dot{Q}^* = \frac{\dot{Q}}{\dot{m} c_p T_1} \quad ; \quad \eta = \frac{\gamma - 1}{\gamma}$$

(a) $$T ds = dh - v\,\cancel{dp}^{\,0 \text{ if } p \text{ is constant}} = c_p dT \text{ for a perfect gas}$$

$$\Rightarrow \int_{T_0}^{T} \frac{dT}{T} = \int_{S_0}^{S} \frac{ds}{c_p}$$

Almost all students completed this successfully

$$\Rightarrow \ln T/T_0 = (S - S_0)/c_p$$

$$\Rightarrow T = T_0 \exp\left(\frac{S - S_0}{c_p}\right)$$

(b) S.F.E.E. in combustor : $\dot{m} c_p (T_3 - T_2) = \dot{Q} = \dot{Q}^* \dot{m} c_p T_1$

$$\Rightarrow T_3 - T_2 = \dot{Q}^* T_1$$

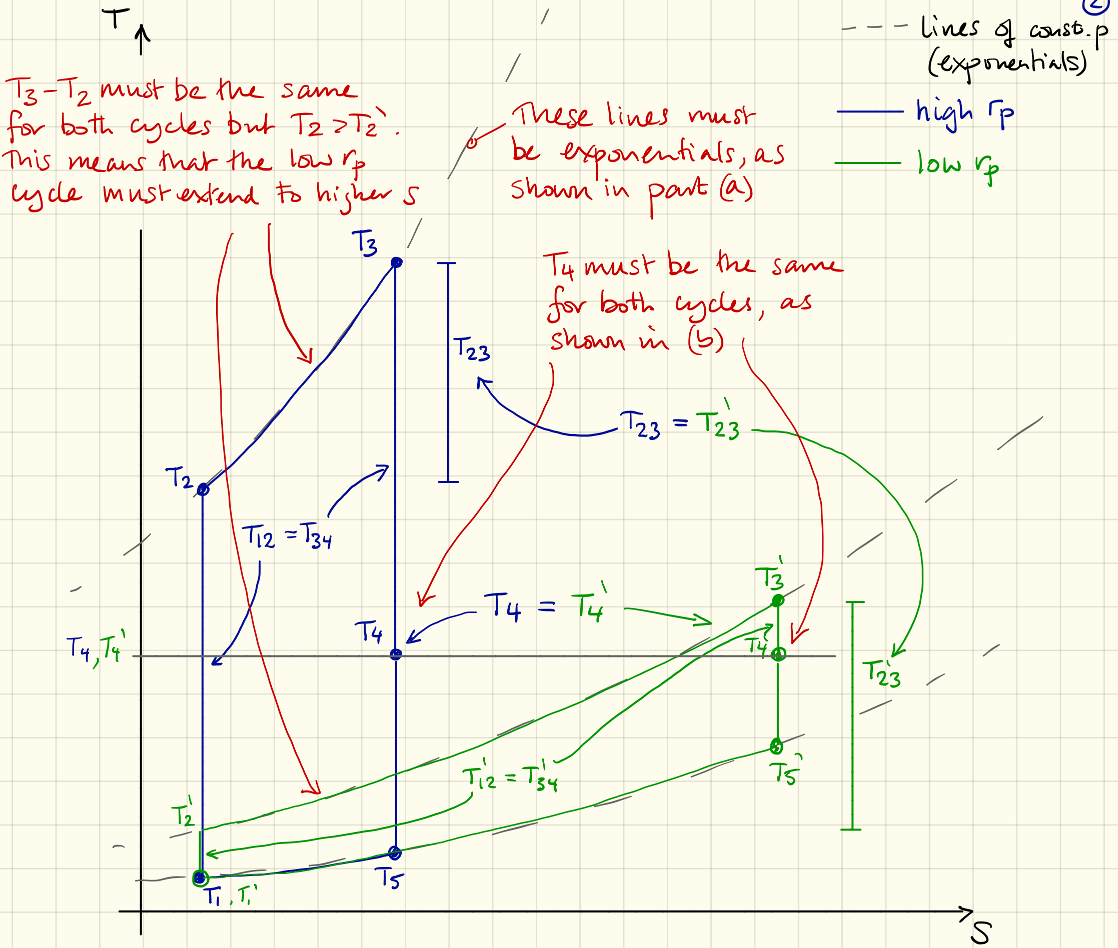
compressor work = turbine work $\Rightarrow T_2 - T_1 = T_3 - T_4$

$$\Rightarrow T_4 = T_3 - T_2 + T_1$$

$$\Rightarrow T_4 = (\dot{Q}^* + 1) T_1$$

Most students completed this but some tried to use isentropic relations and made no progress.

2



Main points to note

- lines of constant p are exponential (part a)
- T_4 does not depend on the compression ratio (part b)
- $T_{12} = T_{34}$ for both cycles but is larger when r_p is larger
- T_{23} does not depend on the compression ratio.

There were some excellent answers but most students answered this badly. The most common error was to assume that the entropy change $S_3 - S_2$ equals $S_3' - S_2'$, even though this is incompatible with $T_4' = T_4$ and $(T_2 - T_1) = (T_3 - T_4)$ for both cycles. For 8 marks students needed to be more careful.

(3)

(d) Isentropic relation across nozzle $\Rightarrow \dfrac{T_4}{T_5} = \left(\dfrac{p_4}{p_5}\right)^{\eta}$

This involves V_5^2 so must use S.F.E.E. Many students did not realise this

Assume K.E. at (4) is negligible.

S.F.E.E. within nozzle $\Rightarrow \dot{m} c_p (T_4 - T_5) = \frac{1}{2}\dot{m} V_5^2$

$$\boxed{M^2 = \frac{V_5^2}{\gamma R T}}$$

at exit (i.e. point 5)

$$\Rightarrow c_p (T_4 - T_5) = \frac{1}{2} M^2 \gamma R T_5$$

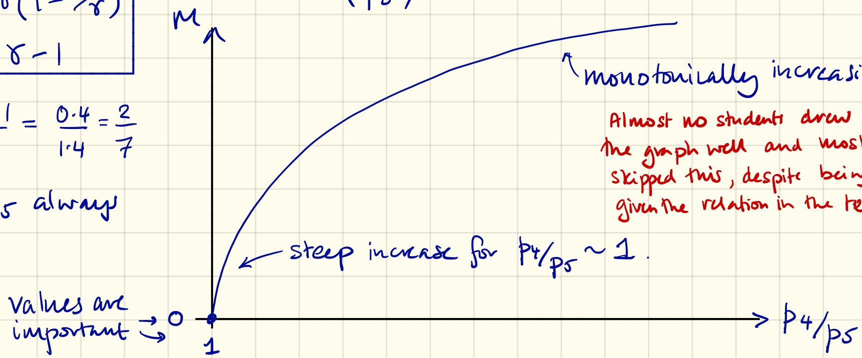
$$\Rightarrow \frac{T_4}{T_5} - 1 = \frac{1}{2} M^2 \frac{\gamma R}{c_p}$$

$$\boxed{\begin{aligned}\frac{\gamma R}{c_p} &= \frac{\gamma (c_p - c_v)}{c_p} \\ &= \gamma (1 - 1/\gamma) \\ &= \gamma - 1\end{aligned}}$$

$$\Rightarrow \left(\frac{p_4}{p_5}\right)^{\eta} - 1 = \frac{1}{2} M^2 (\gamma - 1)$$

$$\eta = \frac{\gamma - 1}{\gamma} = \frac{0.4}{1.4} = \frac{2}{7}$$

$p_4 > p_5$ always



Almost no students drew the graph well and most skipped this, despite being given the relation in the text.

(e) $$\frac{p_4}{p_1} = \left(\frac{p_4}{p_3} \cdot \cancel{\frac{p_3}{p_2}}^{\,1} \cdot \frac{p_2}{p_1}\right) = \left(\frac{T_4}{T_3}\right)^{1/\eta} r_p$$

$$\left.\begin{aligned} T_4 &= T_1 (1 + \dot{Q}^*) \\ T_3 &= T_2 + T_1 \dot{Q}^* \end{aligned}\right\} \text{from (b)}$$

$$= \left(\frac{T_2}{T_1} + \dot{Q}^*\right) T_1$$

$$= (r_p^{\eta} + \dot{Q}^*) T_1$$

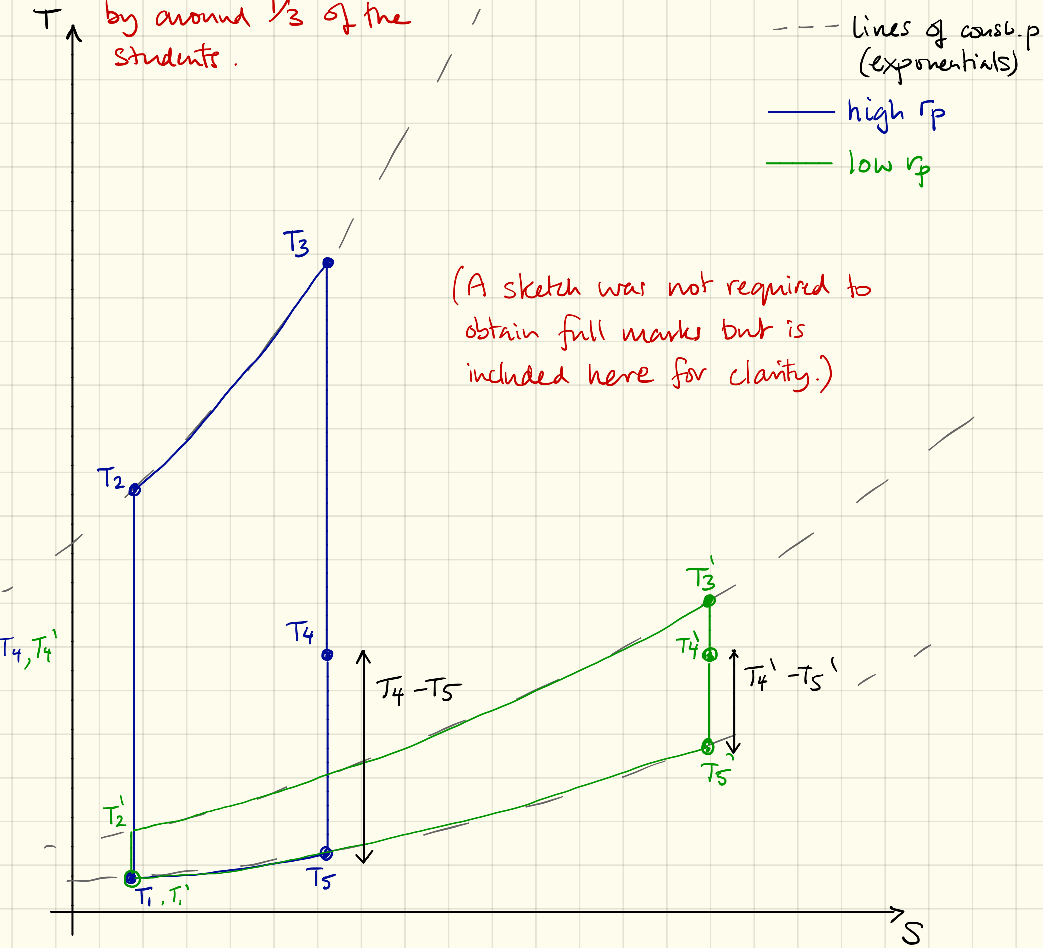
$$= \left(\frac{1 + \dot{Q}^*}{r_p^{\eta} + \dot{Q}^*}\right)^{1/\eta} r_p \longrightarrow r_p \text{ for } \dot{Q}^* \gg 1$$

check: if $r_p = 1$, $p_4/p_1 = 1$

if $\dot{Q}^* = 0$, $p_4/p_1 = 1$

Around 10% of the students answered this well. The remainder did not.

(f) This was well answered by around 1/3 of the students.



(A sketch was not required to obtain full marks but is included here for clarity.)

By the SFEE, $\frac{1}{2}V^2 = h_4 - h_5 = c_p(T_4 - T_5)$

$T_4 - T_5$ is greater for the high p. ratio cycle, so V^2 is higher.
$T_4 - T_5$ is greater because the entropy of the gas in the exhaust is lower, so the gas can drop to a lower temperature in the exhaust when it reaches atmospheric pressure.

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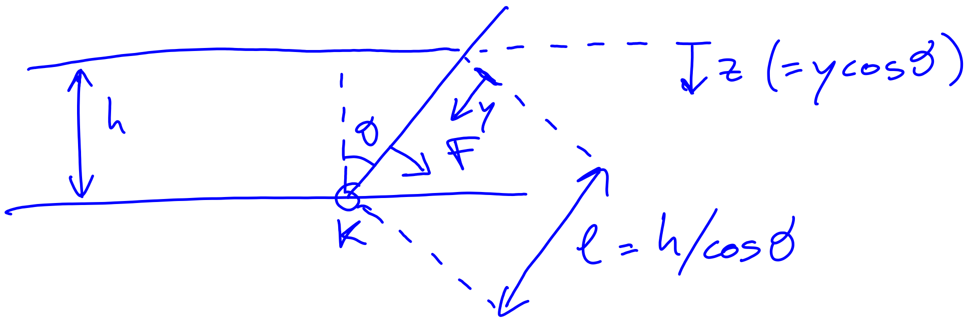
4



a) $F = \int_0^\ell \rho g z\, dy = \rho g \cos\theta \int_0^\ell y\, dy = \frac{1}{2}\rho g \ell^2 \cos\theta = \frac{\rho g h^2}{2\cos\theta}$

F acts $\ell/3$ above pivot $\Rightarrow M = F \cdot \frac{\ell}{3} = \frac{\rho g h^3}{6\cos^2\theta}$ //

(alternatively, $M = \int_0^\ell \rho g z\, dy (\ell - y) = \rho g \cos\theta \int_0^\ell y(\ell-y)\, dy$)

b) Spill $\rightarrow \ell = L$; $h = \frac{\sqrt{3}}{2}\ell \rightarrow \cos\theta = \frac{\sqrt{3}}{2}$, $\theta = \pi/6$ $(30°)$

$M = \frac{\rho g (\sqrt{3}/2\, L)^3}{6(\sqrt{3}/2)^2} = K\theta = K\pi/6$; $K = \frac{\sqrt{3}}{2\pi}\rho g L^3$ //

5

a) From continuity: $V_1 D_1^2 = V_2 D_2^2$, $V_2 = \left(\frac{D_1}{D_2}\right)^2 V_1$ //

b) Bernoulli $p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$

$p_2 = p_1 + \frac{1}{2}\rho \left[V_1^2 - V_2^2\right] = p_1 + \frac{1}{2}\rho V_1^2\left[1 - \left(\frac{D_1}{D_2}\right)^4\right]$ //

c) Bernoulli holds, so Pitot pressure is the same at '1' and '2' $\longrightarrow \Delta h = 0$ //

6

a) Continuity

$$U_\infty \cos\alpha_1 L = \int_{-L/2}^{L/2} v \cos\alpha_2 \, dy = \int_{-L/2}^{L/2} \Big(V - \underbrace{\cancel{\delta V \cos\frac{2\pi y}{L}}}_{\text{integrates to } \emptyset}\Big) \cos\alpha_2 \, dy$$

$$= V \cos\alpha_2 L$$

$$V = \frac{\cos\alpha_1}{\cos\alpha_2} U_\infty$$

b) Bernoulli:

$$p_a + \frac{1}{2}\rho U_\infty^2 = p_a + \frac{1}{2}\rho (V + \delta V)^2$$

$$U_\infty = V + \delta V \longrightarrow \delta V = \left[1 - \frac{\cos\alpha_1}{\cos\alpha_2}\right] U_\infty$$

c) SFME in x

$$F_x = -\rho U_\infty^2 L \cos^2\alpha_1 + \rho \int_{-L/2}^{L/2} (v\cos\alpha_2)(v\cos\alpha_2) \, dy$$

$$= \rho\left[-U_\infty^2 L \cos^2\alpha_1 + \cos^2\alpha_2 \int_{-L/2}^{L/2} \Big(V^2 - 2V\delta V \underbrace{\cos\frac{2\pi y}{L}}_{\text{integrates to } \emptyset} + \delta V^2 \underbrace{\cos^2\frac{2\pi y}{L}}_{\text{integrates to } 1/2}\Big) dy\right]$$

$$= \rho U_\infty^2 L \left[\cancel{-\cos^2\alpha_1} + \cos^2\alpha_2 \left(\frac{\cancel{\cos^2\alpha_1}}{\cos^2\alpha_2} + \frac{1}{2}\left[1 - \frac{\cos\alpha_1}{\cos\alpha_2}\right]^2\right)\right]$$

$$= \frac{1}{2}\rho U_\infty^2 L \left[\cos\alpha_2 - \cos\alpha_1\right]^2$$

Force on flow is $\Rightarrow$, force on aerofoil is $\Leftarrow$

$$\alpha_1 = \cos^{-1}(3/5), \alpha_2 = \cos^{-1}(4/5) \rightarrow F_x = 0.04 \times \frac{1}{2}\rho U_\infty^2 L$$

d) SFME in y

$$F_y = -\rho(U_\infty \cos\alpha_1)(U_\infty \sin\alpha_1)L + \int_{-L/2}^{L/2} (v\cos\alpha_2)(-v\sin\alpha_2)\,dy$$

$$= -\rho U_\infty^2 L\cos\alpha_1\sin\alpha_1 - \cos\alpha_2\sin\alpha_2 \underbrace{\int_{-L/2}^{L/2} v^2\,dy}_{\text{as in (c)}}$$

$$= -\rho U_\infty^2 L\left[\cos\alpha_1\sin\alpha_1 + \cos\alpha_2\sin\alpha_2\left(\frac{\cos^2\alpha_1}{\cos^2\alpha_2} + \frac{1}{2}\left\{1 - \frac{\cos\alpha_1}{\cos\alpha_2}\right\}^2\right)\right]$$

$$= -\rho U_\infty^2 L\left[\frac{1}{2}\sin 2\alpha_1 + \tan\alpha_2\left(\cos^2\alpha_1 + \frac{1}{2}\left\{\cos\alpha_2 - \cos\alpha_1\right\}^2\right)\right]$$

$$= -\frac{1}{2}\rho U_\infty^2 L\left[\sin 2\alpha_1 + \tan\alpha_2\left(2\cos^2\alpha_1 + \left\{\cos\alpha_2 - \cos\alpha_1\right\}^2\right)\right] \,//$$

Force on flow is $\Downarrow$, force on aerofoil is $\Uparrow$ //

$\alpha_1 = \cos^{-1}(3/5)$, $\alpha_2 = \cos^{-1}(4/5)$ $\longrightarrow$ $F_y = -1.53 \times \frac{1}{2}\rho U_\infty^2 L$ //

SECTION B

7 (**short**)

(a) A train is traveling forwards with velocity v. A child throws a tennis ball towards the oncoming train with a horizontal speed u relative to the ground. Assuming the collision is elastic, and the train is far heavier than the ball, at what speed v_B does the ball rebound? [4]

The collision is elastic, so the coefficient of restitution is $e = 1$. i.e. relative speed equal before and after collision.

The train is much heavier than the ball, so it's speed does not change significantly during the collision.

Combining these points:

$$v + u = v_B - v$$

and hence

$$v_B = u + 2v.$$

(b) Three balls of masses $m_1 \gg m_2 \gg m_3$ are vertically stacked at a height h, and then dropped to the ground, as shown in Fig. 1. Assuming that the balls start with small initial separations, and that all collisions occur elastically, find the height to which the mass m_3 rebounds. (You may ignore the radius of the balls.) [6]

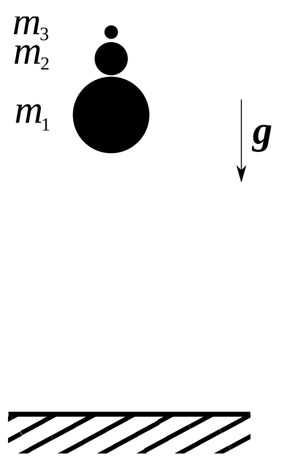


Fig. 1

Conservation of energy shows the balls all reach the floor with velocity $u = -\sqrt{2gh}$, where negative denotes down.

m_1 collides elastically with the floor, giving an upward velocity $v_1 = \sqrt{2gh}$.
m_2 collides elastically with the rising m_1. Using the above result, its exit upward velocity is $v_2 = 3\sqrt{2gh}$.
m_3 collides elastically with m_2.Using the above result, its upward velocity is $v_3 = 7\sqrt{2gh}$.
The ball m_3 rises until all its kinetic energy turns to GPE, giving $H = \frac{1}{2}v^2/g = 49h$.

8 (**short**) An open topped cart is moving at velocity v along a horizontal path, without any friction. It is raining, and the cart is filled by a mass Ω of water each second, as shown in Fig. 2.

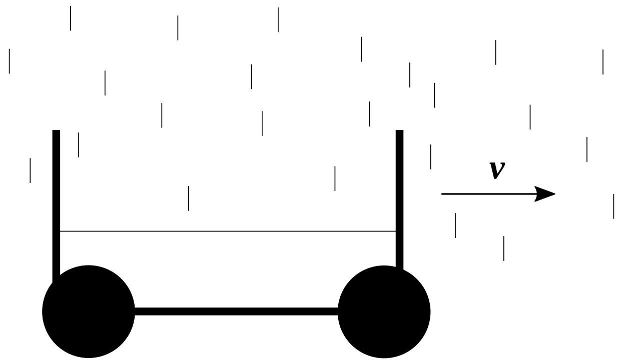


Fig. 2

(a) Find the horizontal force F that must be applied to the cart to maintain its velocity. [4]

At a time t, the mass of the cart is $m_0 + \Omega t$ and has momentum

$$p = (m_0 + \Omega t)v.$$

Newton's second law:

$$F = \frac{dp}{dt} = \Omega v.$$

(b) Find the power required to maintain the cart's velocity. [3]

Power of the driving force is

$$P = Fv = \Omega v^2$$

(c) Find the efficiency with which this power is converted to kinetic energy. Where does the balance of the energy go? [3]

Kinetic energy of the cart is

$$KE = \frac{1}{2}(m_0 + \Omega t)v^2$$

so

$$\frac{dKE}{dt} = \frac{1}{2}\Omega v^2$$

i.e. the process is 50% efficient - half the power of the driving force goes to kinetic energy, and half is lost as heat in the inelastic collision between the cart and the incoming rain.

9 (**long**) A simple seesaw consists of a uniform bar of length $2l$ and mass m with a pivot at its center that allows it to rotate to form an angle θ with the horizontal. Two children, of masses m and $2m$, sit on the ends of the seesaw, with the heavier child touching the ground, and the seesaw making an initial angle θ_0 as shown in Fig. 3. The heavier child then applies an impulse P vertically upwards to launch the seesaw into motion.

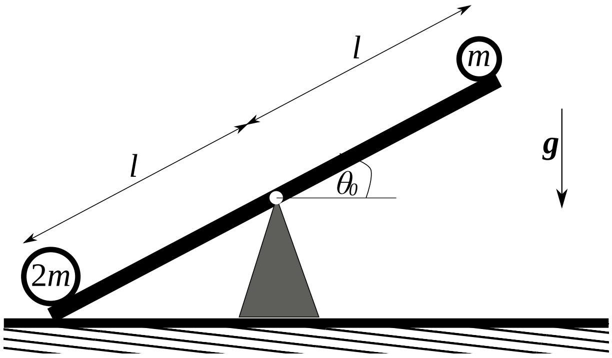


Fig. 3

(a) Show that angular velocity, $\dot{\theta}_0$, of the seesaw directly after the impulse is: [5]

$$\dot{\theta}_0 = -\frac{3P\cos\theta_0}{10ml}.$$

Moment of inertia about the pivot is $I = \frac{1}{12}m(2l)^2 + 2ml^2 + ml^2 = \frac{10}{3}ml^2$.
Angular impulse about pivot is $Pl\cos\theta_0$.
Conservation of angular momentum about pivot

$$-Pl\cos\theta_0 = \frac{10}{3}ml^2\dot{\theta} \quad\rightarrow\quad \dot{\theta}_0 = -\frac{3P\cos\theta_0}{10ml}$$

(b) Find the reaction impulse applied to the rod by the pivot, giving you answer in horizontal and vertical components. [5]

Take center of mass a distance d from from pivot.

$$m(l+d) + md = 2m(l-d) \quad\rightarrow\quad d = l/4.$$

Instantaneous center is at pivot, so velocity of CoM is perpendicular to the rod with magnitude:

$$v = -\frac{l}{4}\dot{\theta}_0$$

Conservation of momentum vertically and horizontally give the vertical and horizontal reaction impacts at the pivot:

$$R_v + P = -4m\frac{l}{4}\dot{\theta}_0\cos\theta_0 = P\left(\frac{3}{10}\cos^2\theta_0 - 1\right)$$

 (cont.

and

$$R_h = 4m\frac{l}{4}\dot{\theta}_0 \sin\theta_0 = -P\frac{3}{10}\cos\theta_0 \sin\theta_0 \quad \text{(i.e. to the left)}$$

During the motion of the seesaw, sliding within the pivot joint causes a frictional torque τ of constant magnitude. When either child reaches the ground, they apply a vertical impulse that reverses their velocity.

(c) Find the equation of motion for θ, and describe a forward Euler numerical integration scheme to predict the seesaw's subsequent motion over multiple oscillations. [8]

Moment around the pivot is

$$G = mgl\cos\theta - \tau\mathrm{sgn}(\dot{\theta}).$$

Using $I\ddot{\theta} = G$, the equation of motion is

$$\frac{10}{3}ml^2\ddot{\theta} = mgl\cos\theta - \tau\mathrm{sgn}(\dot{\theta}).$$

Simple Forward Euler numerical integration scheme:

(1) Initialization: set $t = 0$, $\theta = \theta_0$ and $\dot{\theta} = \dot{\theta}_0$ using initial data from just after impulse.

(2) Compute current $\ddot{\theta}$ from the equation of motion, using the current values of θ and $\dot{\theta}$.

(3) Update the current variables by one time step, $\theta \to \theta + \dot{\theta}\Delta t$, $\dot{\theta} \to \dot{\theta} + \ddot{\theta}\Delta t$, $t \to t + \Delta t$.

(4) If $\theta > \theta_0$ or $\theta > -\theta_0$, collision with ground detected, update $\dot{\theta} \to -\dot{\theta}$.

((4b) Possible extra if here for changing sign of τ so it always opposes $\dot{\theta}$, but unnecessary if you have explicitly used a sgn *function in definition of equation of motion)*

(5) If $\theta > -\theta_0$, lighter child is touching ground. Update count $\to$ count + 1.

(6) Goto (2), and loop until finish time reached.

(d) Sketch a graph of $\theta(t)$ for the case where the lighter child touches the ground twice. [6]

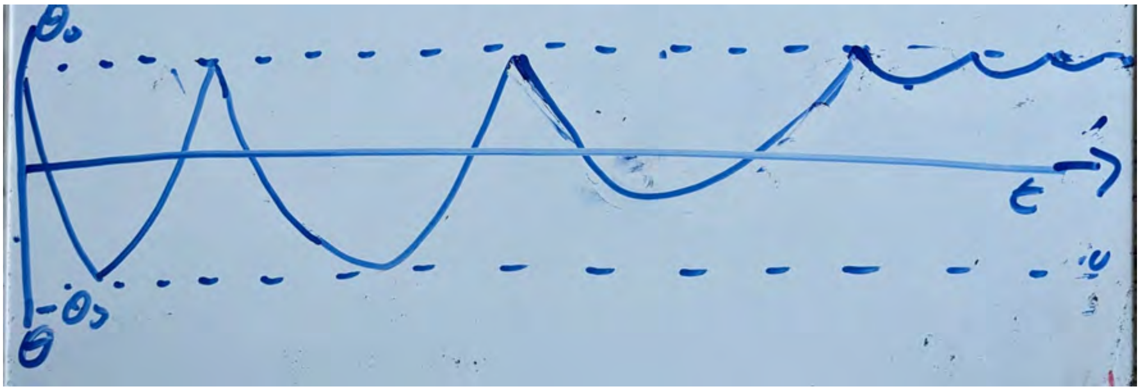
Key points

(1) Seesaw starts at θ_0 with a finite negative velocity (slope).

(2) Seesaw slows down as it goes from positive to negative, and speeds up (though less due to τ) as it goes from negative to positive.

(3) Velocity reverses at each contact with ground, i.e. discontinuous reversal of slope.

(4) Long time asymptote has diminishing amplitude and time period, like a damped bouncing ball.



(e) The initial impulse is chosen such that the lighter child just makes contact with the ground. Show that the required value of the first impulse is: [6]

$$P^2 = \frac{40m}{3\cos^2\theta_0}\left(mgl\sin\theta_0 + \tau\theta_0\right).$$

During this part of the motion, $\dot{\theta}$ is negative so the frictional torque and gravity both act in the same sense:

$$\frac{10}{3}ml^2\ddot{\theta} = mgl\cos\theta + \tau.$$

Multiply by $\dot{\theta}$ and integrate dt*, to get*

$$\frac{5}{3}ml^2\dot{\theta}^2 - mgl\sin\theta - \tau\theta = c.$$

where c is a constant of integration that is analogous to the total energy. Equating this constant at the start ($\theta = \theta_0$ and $\dot{\theta}_0 = \frac{3P\cos\theta_0}{10ml}$) and end ($\dot{\theta} = 0$, $\theta = -\theta_0$) of the motion gives

$$\frac{5}{3}ml^2\frac{9P^2\cos^2\theta_0}{100m^2l^2} - mgl\sin\theta - \tau\theta = mgl\sin\theta + \tau\theta.$$

Solving for P^2 gives the stated result.

Q. 10

(a) $\vec{v} = (r_0 e^{\beta t} \omega)\,\hat{e}_\theta + (\beta r_0 e^{\beta t})\,\hat{e}_r$

$\vec{a} = (r_0 e^{\beta t} \times 0 + 2 \times \beta r_0 e^{\beta t} \times \omega)\,\hat{e}_\theta$

$+ (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t} \omega^2)\,\hat{e}_r$

$= (\beta^2 r_0 e^{\beta t} - r_0 e^{\beta t} \omega^2)\,\hat{e}_r$

$+ (2\beta r_0 e^{\beta t} \omega)\,\hat{e}_\theta$

(b) Radial component of acceleration is zero if $\beta^2 = \omega^2$

$\beta = \pm\,\omega.$

(a) From the databook:

$$\zeta = 0.5$$

logarithmic decrement

$$= \frac{2\pi \times 0.5}{\sqrt{1-0.25}} = 3.63$$

(b)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{10^{-6}}} = \cancel{4.4}\,1.41 \times 10^{4} \text{ rad/sec}$$

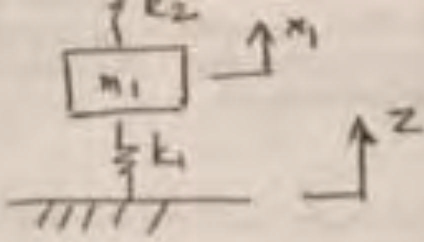
$$\frac{Y}{\omega^2 X} = \frac{\left(1/\omega_n\right)^2}{\left(-\omega/\omega_n\right)^2 + 2i\zeta\omega/\omega_n + 1}$$

$$\frac{Y}{10^{-2}} = \frac{1}{-\omega^2 + 2i\zeta\omega_n\omega + \omega_n^2}$$

$$= \frac{1}{2\times10^{8} - (2\pi\times1000)^2 + 2i\times0.5\times\cancel{1}\times4\times10^{4} \times 2\pi\times1000}$$

$$= \frac{1}{160521582 + 8.85\times10^{4}\, i}$$

$$\text{mag}(Y) = 6.23\times10^{-11}\text{ m}$$



(a)
$$m\ddot{x}_1 + k_1(x_1 - z) + k_2(x_1 - x_2) = 0.$$
$$m\ddot{x}_2 + k_2(x_2 - x_1) = 0.$$
$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 z \\ 0 \end{bmatrix}$$

(b)
$$K - M\omega^2 = 0$$
$$\begin{bmatrix} k_1+k_2 - m\omega_n^2 & -k_2 \\ -k_2 & k_2 - m\omega_n^2 \end{bmatrix} = 0.$$
$$(k_1+k_2 - m\omega_n^2)(k_2 - m\omega_n^2) - k_2^2 = 0.$$
$$(3k - m\omega_n^2)(k - \omega_n^2 m) - k^2 = 0$$
$$2k^2 - 4km\omega_n^2 + m^2\omega_n^4 = 0.$$
$$\omega_n^2 = \frac{4km \pm \sqrt{16k^2m^2 - 8k^2m^2}}{2m^2}$$
$$\omega_n^2 = (2 \pm \sqrt{2})\sqrt{\frac{k}{m}}$$

(c) $k = 2\times10^5$ N/m , $m = 10^5$ kg

$$\omega_n^2 = (2\sqrt{2} \pm 2)\frac{k}{m} \quad \text{or} \quad \omega_{n1} = 0.9\ \cancel{\text{Hz}}\ \text{rad/s}, \quad \omega_{n2} = 2.2\ \cancel{\text{Hz}}\ \text{rad/s}$$

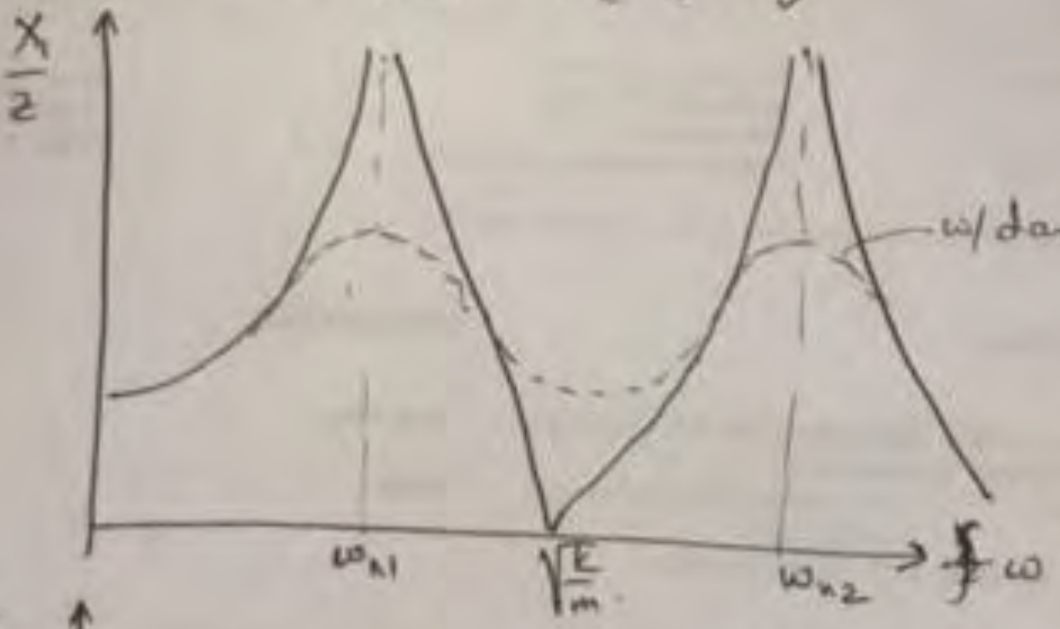
(c)
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} k_2 - m\omega^2 & k_2 \\ k_2 & k_1 + k_2 - m\omega^2 \end{bmatrix}\begin{bmatrix} k_1 Z \\ 0 \end{bmatrix}$$

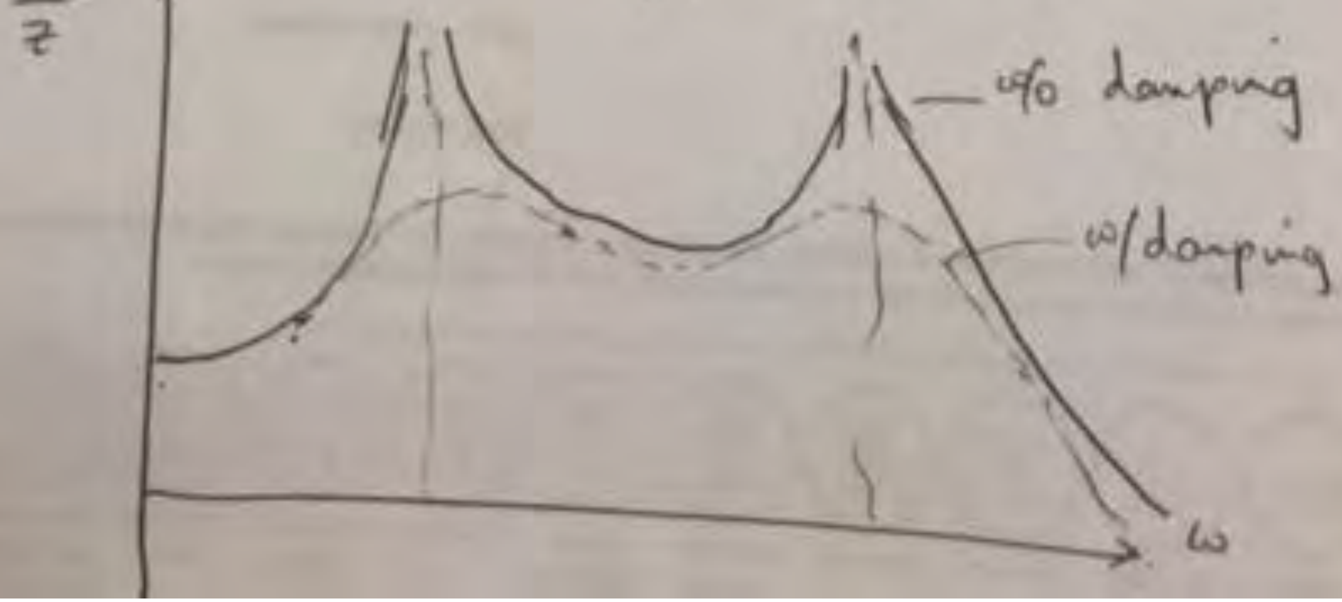
where

$$\Delta = (k_1 + k_2 - m\omega^2)(k_2 - m\omega^2) - k_2^2$$

$$\frac{X_1}{Z} = \frac{(k_2 - m\omega^2)k_1}{m^2(\omega_{n1}^2 - \omega^2)(\omega_{n2}^2 - \omega^2)}$$

$$\frac{X_2}{Z} = \frac{k_1 k_2 \cancel{Z}}{m^2(\omega_{n1}^2 - \omega^2)(\omega_{n2}^2 - \omega^2)}$$





(d). $z = 1\,cm$, $\omega = 1\,rad/s$.

$$x_1 = \frac{k_1 k_2}{m^2}\left[\frac{(1 - m\omega^2/k_2)}{(\omega_{n1}^2 - \omega^2)(\omega_{n}^2 - \omega^2)}\right].$$

$$= 2\times 4\left[\frac{\left(1 - \frac{1}{2}\right)}{(12+8\sqrt{2}-1)(12-8\sqrt{2}-1)}\right].$$

$$= -4/7 \text{ cm}.$$

$$x_2 = 2\times 4 \times \frac{1}{(-7)} = -8/7 \text{ cm}.$$

The motion of x_2 exceeds 1cm ground/critical displacement.