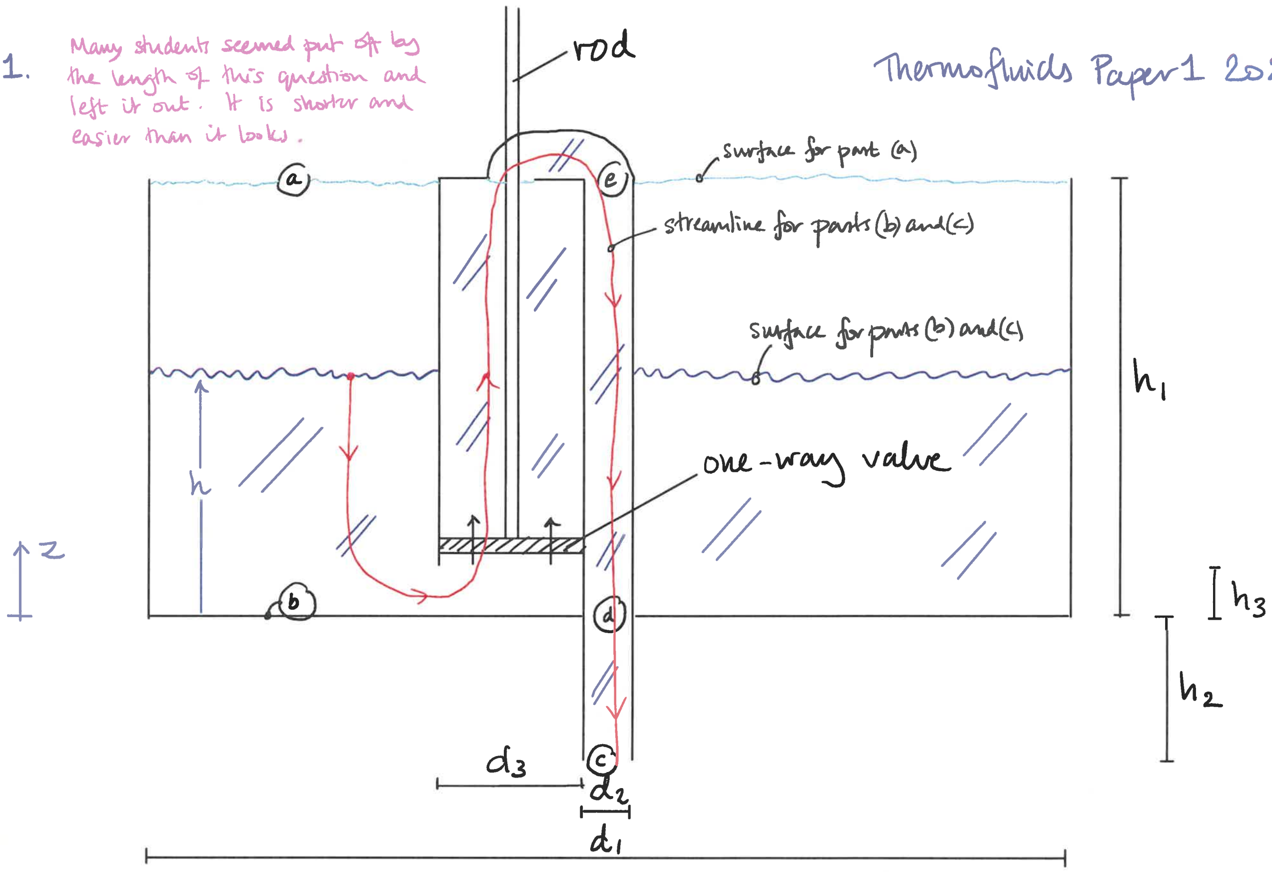
1. Many students seemed put off by the length of this question and left it out. It is shorter and easier than it looks.

Thermofluids Paper 1 2025



a) The tank and cylinder fill with water but the J-shaped pipe still contains air. The pressures are hydrostatic. The gauge pressures are:

$p_a = p_c = p_d = p_e = 0$ and $p_b = \rho g h_1$ Most students answered this well. [2]

b) Assume no mechanical energy losses so that Bernoulli's equation is valid along any streamline in the fluid. The fluid has general height h above the bottom of the tank. The gauge pressures are:

$p_a = p_c = 0$; $p_b = \rho g h$; $p_d = -\rho g h_2$; $p_e = -\rho g (h_1 + h_2)$ [2]

Few students answered this well, even though it is similar to an examples paper question. The result seems counter-intuitive, suggesting that most students are still relying on their intuition to answer simple questions.

(c) Consider a streamline from the water surface at height h with negligible velocity (because $d_1 >> d_2$) and $p = 0$ to the pipe exit at height $-h_2$ with velocity V_c and $p = 0$. By Bernoulli: [4]

$$p + \rho g h = p + \tfrac{1}{2}\rho V_c^2 - \rho g h_2$$

$$\Rightarrow \quad V_c^2 = 2g(h + h_2)$$

$$\Rightarrow \text{volumetric flowrate} = \frac{\pi d_2^2}{4}\sqrt{2g(h+h_2)}$$

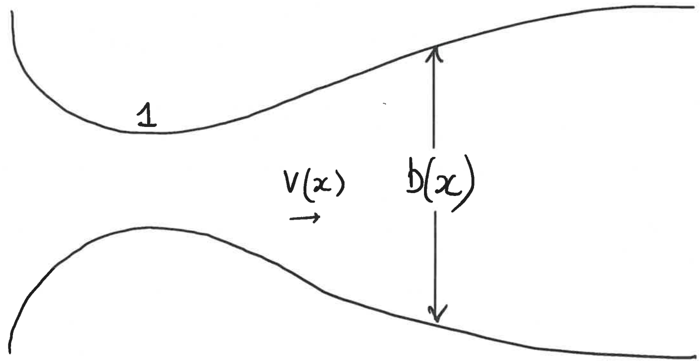
Most students who attempted this answered it well. Many students had, however, abandoned the question at part (b).

(d) Drill a hole half-way down the rod. As the water level in the tank drops, the water pressure at the hole drops. When this drops below atmospheric pressure, air will be sucked into the cylinder and the syphon effect will stop. [2]

This required some lateral thinking. Around 5% of students answered this correctly and, of these, some had not answered the previous parts well.

M. Juniper 2025

2.



There is a nice analogy between the two long questions in this exam, in particular between $Fr^2 \equiv V_1^2/gh$ in Q2 and $V_1^2/2c_pT_1$ in Q5. Early versions of the questions made this clear but the practicalities of making questions that can be solved in 30 minutes prevented this from enduring to the final versions. Interested students should do compressible flow in 3rd year (3A3).

Most students answered this well

(a) [2] The streamlines are straight (i.e. no curvature) so the flow is not accelerating perpendicular to the streamlines. This means that the pressure gradient must exactly balance the gravitational body force ⇒ the pressure gradient is hydrostatic

(b) [4] If friction with the channel walls is neglected and there is no hydraulic 'jump' and no turbulence then there is no loss of mechanical energy. This means that Bernoulli's equation can be applied throughout the flow. For simplicity, consider the surface streamline, on which $p = p_a$:

$$p_a + \tfrac{1}{2}\rho V^2 + \rho g h = \text{const.} \Rightarrow V^2 + 2gh = \text{const.}$$

This was mostly well answered, although many students did not write the complete justification above

(c) [6] By Bernoulli (part b), $\frac{1}{2}V_1^2 + gh_1 = gh_0$ throughout the flow.

At section 1, $Fr = 1 \Rightarrow V_1^2 = gh_1 \Rightarrow \frac{3}{2}gh_1 = gh_0$

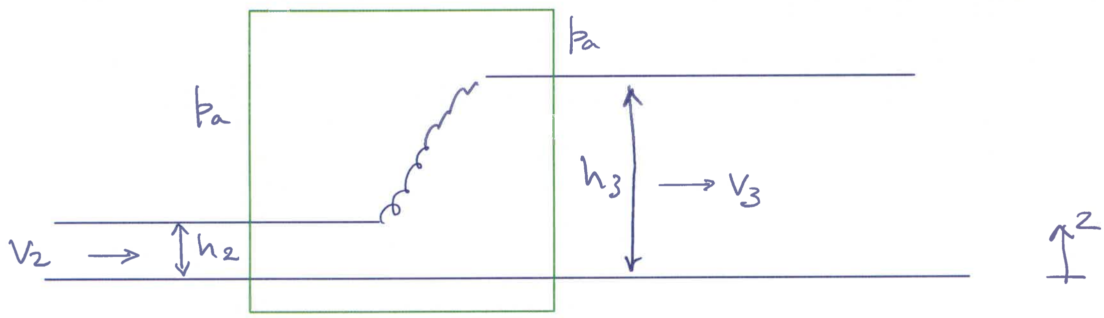
$$\Rightarrow h_1 = \frac{2}{3}h_0 \quad \text{and} \quad V_1 = \left(\frac{2}{3}gh_0\right)^{1/2}$$

This was answered well by around 50% of students. These students also answered part (d) well.

$$\text{Volumetric flowrate} = b_1 h_1 V_1 = \left(\frac{2}{3}\right)^{3/2} g^{1/2} h_0^{3/2} b_1$$

(d) [2] $V^2 + 2gh = 2gh_0$ continues to apply downstream. If h continues to decrease then V continues to increase. The maximum attainable speed is achieved when $h \to 0$ and is $(2gh_0)^{1/2}$.

(e)



i) [2] conservation of mass at given b ⇒ $V_2 h_2 = V_3 h_3$

$$\Rightarrow \frac{V_2^2 h_2^3}{g h_2} = \frac{V_3^2 h_3^3}{g h_3} \Rightarrow Fr_2^2 = H^3 Fr_3^2 \quad \text{where } H \equiv \frac{h_3}{h_2}$$

Most students answered this well.

ii) Mechanical energy is lost across the jump so Bernoulli cannot be used.

[6] Steady flow momentum equation can be used on control volume shown

forces: $p_2(z) = \rho g(h_2 - z)$ in gauge pressure

$$\Rightarrow F_2 = \int_0^{h_2} p_2(z)\, b\, dz = \frac{1}{2}\rho g h_2^2 b \quad \text{and} \quad F_3 = \frac{1}{2}\rho g h_3^2 b$$

$$\Rightarrow \dot{m} V_2 + \frac{1}{2}\rho g h_2^2 b = \dot{m} V_3 + \frac{1}{2}\rho g h_3^2 b$$

$$\Rightarrow \cancel{\rho}\cancel{b} h_2 V_2^2 + \frac{1}{2}\cancel{\rho} g h_2^2 \cancel{b} = \cancel{\rho}\cancel{b} h_3 V_3^2 + \frac{1}{2}\cancel{\rho} g h_3^2 \cancel{b}$$

$$\Rightarrow \frac{V_2^2}{g h_2} + \frac{1}{2} = \frac{h_3^2}{h_2^2}\left\{\frac{V_3^2}{g h_3^2} + \frac{1}{2}\right\}$$

$$\Rightarrow 2Fr_2^2 + 1 = \left(\frac{h_3}{h_2}\right)^2 \left\{2Fr_3^2 + 1\right\}$$

$$\Rightarrow 2Fr_2^2 + 1 = H^2(2Fr_3^2 + 1)$$

Students who carefully set up a control volume, as shown here, answered it well. Several tried to use their intuition and ad hoc methods inspired by school physics, which ended badly.

This was easy and most students answered it well, even if they had not answered previous sections

(f) $Fr_2^2 = H^3 Fr_3^2$

$2Fr_2^2 + 1 = H^2(2Fr_3^2 + 1)$

$H = 1$ and $Fr_2^2 = Fr_3^2$ should be a solution. It is, by inspection. [2]

Solve for H in terms of Fr_2^2. Start by eliminating Fr_3^2:

$$2Fr_2^2 + 1 = H^2\left(2\frac{Fr_2^2}{H^3} + 1\right) = 2\frac{Fr_2^2}{H} + H^2$$

$$\Rightarrow \quad H^3 - H(2Fr_2^2 + 1) + 2Fr_2^2 = 0$$

We know that $H=1$ is a solution, so factorize it out:

$$\Rightarrow \quad (H-1)(H^2 + H - 2Fr_2^2) = 0$$

Few students performed this factorization successfully.

$$\Rightarrow \quad H = 1 \text{ or } H = \frac{-1 \pm (1 + 8Fr_2^2)^{1/2}}{2}$$

the negative root is un-physical.

If $Fr_2^2 = 3$ then $H = 2$

Around 20% of students wrote excellent solutions to this question. [6]

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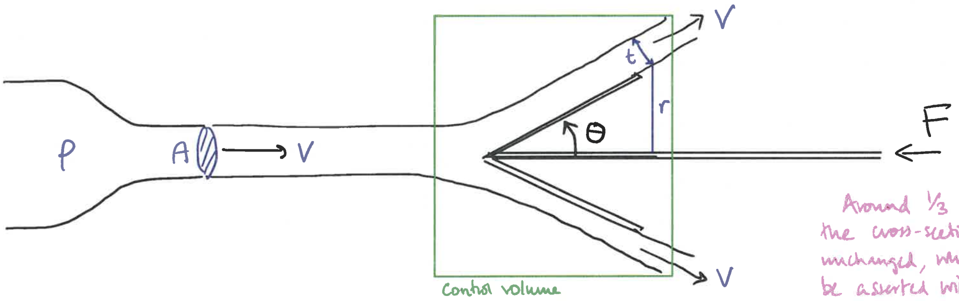
3.



Around 1/3 of students asserted that the cross-sectional area would remain unchanged, which is true but which cannot be asserted without this energy argument

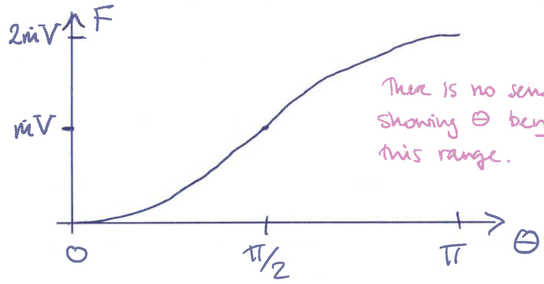
(a) The pressure is uniform around the jet before and after it hits
[2] the cone. Assuming no loss of mechanical energy Bernoulli's equation requires $V_{out} = V_{in} = V$.

(b) Consider a control volume enclosing the cone. The pressure
[4] is uniform so the pressure forces cancel. If F is the force exerted on the C.V. by the cone then:

$$\dot{m}U = F + \dot{m}U\cos\theta$$

$$\Rightarrow \quad F = \dot{m}V(1-\cos\theta)$$

where $\dot{m} = \rho A V$



There is no sense in showing θ beyond this range.

The model will break down when there is significant turbulent mixing in the jet around the cone. This will certainly happen for angles around $\theta = \pi$ and will start to happen for angles somewhere between $\theta = \pi/2$ and π.

Most students answered this well.

c) By Bernoulli (conservation of energy), the exit velocity is V. If $t \ll r$, the
[4] direction of flow is perpendicular to a cross-section with area $2\pi r t$

By conservation of volume, $AV = 2\pi r t V$

$$\Rightarrow \quad t = \frac{A}{2\pi r}$$

Surprisingly few students attempted this question, even though it is an easy application of conservation of mass.

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4. $pv^n = k = \text{const} \Rightarrow \dfrac{p_1}{p_2} = \left(\dfrac{v_2}{v_1}\right)^n$ [2]

a) $n = \gamma \Rightarrow pv^\gamma = \text{const} =$ isentropic

$n = 1 \Rightarrow pv = \text{const} =$ isothermal

$n = 0 \Rightarrow p = \text{const} \Rightarrow$ constant pressure

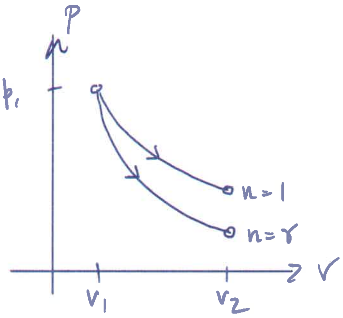
$n \to \infty$ for a constant volume process.

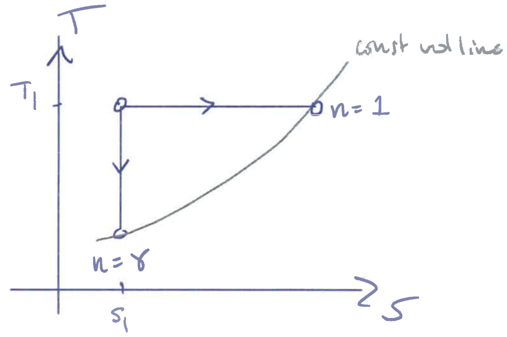
This question was answered well by nearly all students.

b) work done $= \displaystyle\int_{v_1}^{v_2} p\,dv = k\int_{v_1}^{v_2} v^{-n}\,dv = k\left[\frac{v^{1-n}}{1-n}\right]_{v_1}^{v_2} = \frac{k}{1-n}\left\{v_2^{1-n} - v_1^{1-n}\right\} = \frac{1}{1-n}\left\{\frac{kv_2}{v_2^n} - \frac{kv_1}{v_1^n}\right\} = \frac{p_2v_2 - p_1v_1}{1-n}$

$= \dfrac{p_1v_1 - p_2v_2}{n-1}$ [4]

c)





[4]

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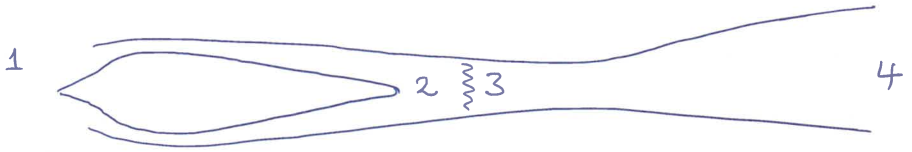
5. Ramjet:



a) SFEE between 1 and 2 : $C_pT_1 + \frac{1}{2}V_1^2 = C_pT_2$ $\quad (V_2^2 \ll V_1^2)$

[2]

$$\Rightarrow \frac{T_2}{T_1} = 1 + \frac{V_1^2}{2C_pT_1}$$

Nearly every student answered this well.

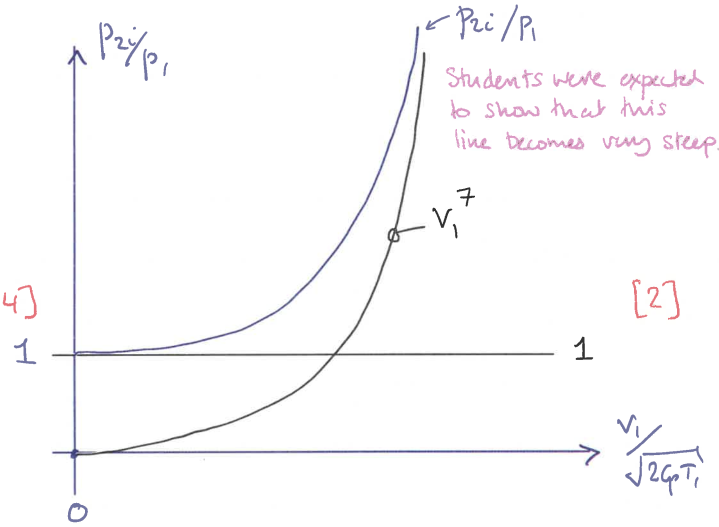
b) reversible and adiabatic

$\Rightarrow$ isentropic

$$\Rightarrow \frac{T}{p^{\frac{\gamma-1}{\gamma}}} = \frac{T}{p^{2/7}} = \text{const}$$

$$\Rightarrow \frac{p_{2i}}{p_1} = \left(\frac{T_2}{T_1}\right)^{7/2} = \left(1 + \frac{V_1^2}{2C_pT_1}\right)^{7/2}$$ [4]

Nearly all students derived the expression correctly but few sketched the graph well.



Students were expected to show that this line becomes very steep.

[2]

c) If $T_1 = 200\text{K}$, $p_1 = 0.03\text{MPa}$, $\frac{V_1}{\sqrt{2C_pT_1}} = 2$ then $T_2/T_1 = 5$ and $\frac{p_{2i}}{p_1} = 5^{7/2}$

$\Rightarrow T_2 = 1000\text{K}$ and $p_{2i} = 0.03\text{ MPa} \times 5^{7/2} = 8.39\text{ MPa}$

$\Rightarrow V_1 = (2C_pT_1)^{1/2}\bar{V}_1 = (2 \times 1005 \times 200)^{1/2} \times 2 = 1268\text{ ms}^{-1}$ [3]

This is very fast, although only 50% faster than the 1950's SR71 Blackbird. It is achievable today but the temperature and pressure are high and will require careful engineering. [1]

Most students obtained correct numerical solutions, although few commented on them. Any sensible comment earned a mark.

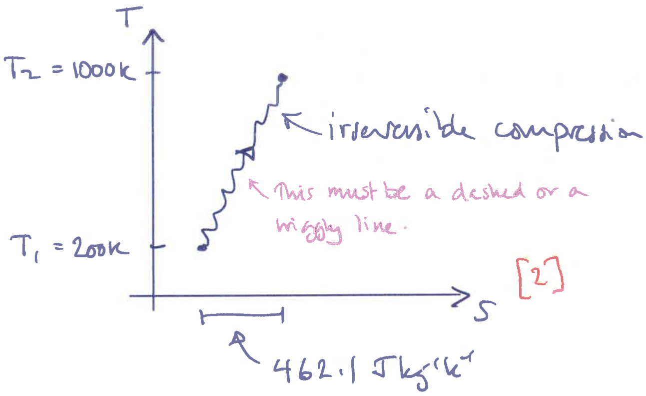
(d) $$s_2 - s_1 = C_p\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = C_p\ln\left(\frac{T_2}{T_1}\right) - R\ln\left\{\alpha\frac{p_{2i}}{p_1}\right\}$$

$$= \underbrace{C_p\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_{2i}}{p_1}\right)}_{\text{This is zero because it corresponds to an isentropic change}} - R\ln\alpha$$

No student spotted the quick way to do this question. Most simply plugged in the numbers from the previous parts and obtained a rounding error.

$$= -R\ln(0.2)$$

$$= 462.1\text{ J kg}^{-1}\text{K}^{-1}$$ [4]



This must be a dashed or a wiggly line.

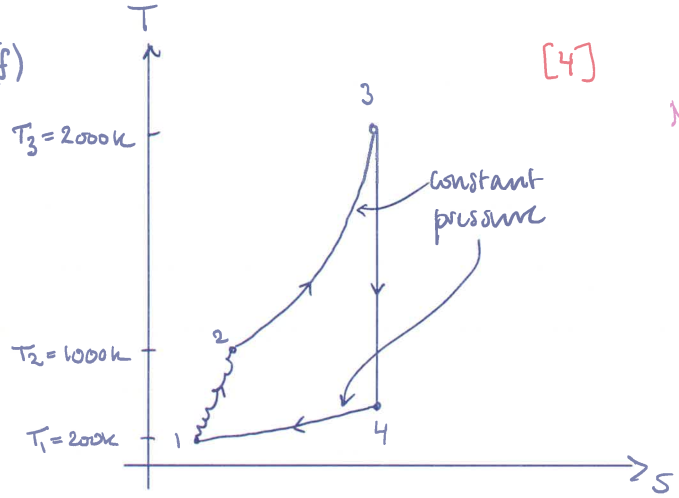
[2]

(e) [2] SFEE from 2 to 3: $\quad c_p(T_3 - T_2) = q$

Nearly every student answered this correctly. Around 10% used 1st law for a system (not a c.v.) and used c_v to obtain 2400 K.

$$\Rightarrow T_3 = T_2 + q/c_p = 1000\,\text{K} + \frac{1.005 \times 10^6}{1005} = 2000\,\text{K}$$

(f) [4]



Most students drew this well.

(g) [6]

$$\frac{T_3}{T_4} = \left(\frac{p_3}{p_4}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\alpha \frac{p_{2i}}{p_1}\right)^{2/7} = \alpha^{2/7}\left(\frac{p_{2i}}{p_1}\right)^{2/7} = \alpha^{2/7}\Big(1 + \underbrace{\frac{V_1^2}{2c_pT_1}}_{4 \text{ from (c)}}\Big) = 0.2^{2/7} \times 5 = 3.157$$

$$\Rightarrow T_4 = T_3 / 3.157$$

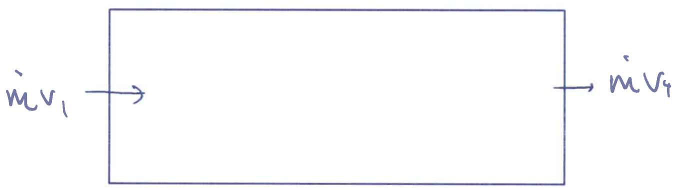
SFEE: $\quad c_pT_4 + \frac{1}{2}V_4^2 = c_pT_3$

$$\Rightarrow V_4^2 = 2c_p(T_3 - T_4) = 2c_pT_3\left(1 - \frac{T_4}{T_3}\right) = 2c_pT_3\left(1 - \frac{1}{3.157}\right)$$

$$= 2 \times 1005 \times 2000 \left(1 - \frac{1}{3.157}\right)$$

$$\Rightarrow V_4 = 1657\,\text{m s}^{-1}$$

Around 25% of students answered this very well.



Consider a control volume moving with the ramjet. The pressure is uniform so:

SFME: $\quad \frac{\text{force}}{\dot{m}} = V_4 - V_1 = 1657 - 1268 = 389\,\text{m/s} = 389\,\frac{\text{Ns}}{\text{kg}}$

M. Juniper 2025

6.

a) $COP = \frac{Q_h}{W}$; $Q_h = Q_C + W$ and $\frac{Q_C}{T_C} = \frac{Q_H}{T_H}$ if reversible.

[2]

$$\Rightarrow Q_h = \frac{T_C}{T_H} Q_H + W$$

$$\Rightarrow Q_h \left(1 - \frac{T_C}{T_H}\right) = W$$

$$\Rightarrow COP_m = \frac{Q_h}{W} = \frac{1}{1 - T_C/T_H} = \frac{T_H}{T_H - T_C}$$

← Nearly all students answered this well.

b) $W_{actual} = \alpha \frac{Q_h}{COP_m}$ $\Rightarrow$ $COP = \frac{Q_h}{W_{actual}} = \frac{COP_m}{\alpha} = \frac{1}{\alpha} \frac{T_H}{T_H - T_C}$

[2]

c) Define W_{chem} as the chemical power from the gas. For the gas boiler, $Q_h = W_{chem}$

[2]

$W_{elec} = \eta W_{chem}$ is the electrical power.

For the heat pump, $Q_h = COP_m \frac{1}{\alpha} W_{actual} = COP_m \frac{\eta}{\alpha} W_{chem}$

These are the same when $\cancel{W_{chem}} = COP_m \frac{\eta}{\alpha} \cancel{W_{chem}}$

$$\Rightarrow \alpha/\eta = COP_m = \frac{T_H}{T_H - T_C}$$

Surprisingly many students made algebraic mistakes here, although they had the correct principle.

$$\Rightarrow \alpha(T_H - T_C) = \eta T_H$$

$$\Rightarrow T_H(\alpha - \eta) = T_C \alpha$$

$$\Rightarrow \frac{T_H}{T_C} = \frac{\alpha}{\alpha - \eta}$$

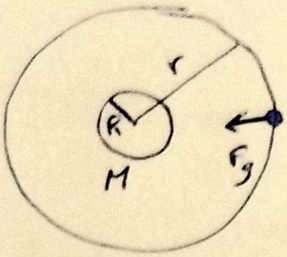
check: As $\eta \to 0$ or $\alpha \to \infty$, cross-over occurs at $T_C/T_H \to 1$. ✓

d) if $T_C = 273K$, $T_H = 273\left(\frac{2}{2-0.6}\right) = 390K = 117$ celcius [2]

This is above the boiling point of water at $P_a = 1$ atm so the heat pump will beat the boiler in all practical applications. The college should also consider the capital cost of installing large radiators or under-floor heating and the running cost of electricity, which is three times greater than gas for the same power. [2]

M. Juniper 2025

Q7 a) The satellite is in circular motion.



$$ma = F_g$$

$$\cancel{m}\omega^2 r = G\frac{M\cancel{m}}{r^2}$$

$$r^3 = \frac{GM}{\omega^2}$$

We need to estimate ω.

1 revolution per day

$$\omega = 2\pi \frac{1}{24 \times 60 \times 60} = 7.27 \cdot 10^{-5} \text{ rad/s}$$

$$\rightarrow r = \sqrt[3]{\frac{GM}{\omega^2}} = 4.22 \cdot 10^{7} \text{ m} = 42{,}200 \text{ km}$$

The altitude h is:

$$h = r - R = 35\,800 \text{ km}.$$

b) The work done is the difference in total energy.

Difference of potential energy:

$$\Delta U = -\frac{GMm}{r} - \left(-\frac{GMm}{R}\right) = GMm\left(\frac{1}{R} - \frac{1}{r}\right)$$

$$\Delta U = 5.30 \times 10^{10} \text{ J}$$

Kinetic energy on orbit

$$E_c = \frac{1}{2} m (r\omega)^2 = 4.71 \times 10^{9} \text{ J}$$

We may take into account the kinetic energy on the Earth before launch.

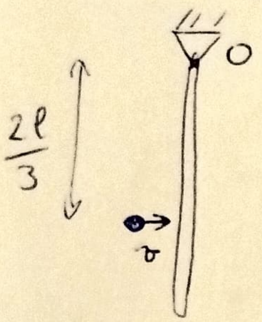
$$\frac{1}{2} m (R\omega)^2 = 1.08 \times 10^{8} \text{ J}$$

But the contribution is not significant.

Work done $\quad W = \Delta U + \Delta E_c = 5.76 \cdot 10^{10} \text{ J}$

Q8

a)



Taking moments about O for the system (bar + projectile).

The reactive impulse on the hinge O does not contribute

(before) (after)

$$m v \frac{2\ell}{3} = \Big[\underbrace{I_g + m\left(\frac{\ell}{2}\right)^2}_{\text{bar, // axis theorem}} + \underbrace{m\left(\frac{2\ell}{3}\right)^2}_{\text{projectile}} \Big] \omega$$

$$I_g = m \frac{\ell^2}{12}$$

$$m v \frac{2\ell}{3} = m\ell^2\omega\left(\frac{1}{12} + \frac{1}{4} + \frac{4}{9}\right) = \frac{7}{9} m \ell^2 \omega$$

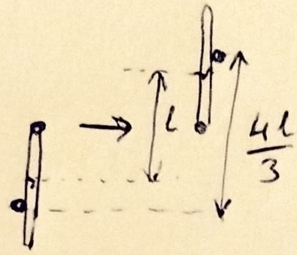
$$\omega = \frac{6}{7} \frac{v}{\ell}$$

b) The kinetic energy after impact must match the change in potential energy.

$$E_c = \frac{1}{2} I \omega^2$$

$$= \frac{1}{2}\left[m \ell^2 \frac{7}{9}\right] \frac{6^2}{7^2} \frac{v^2}{\ell^2}$$

$$= \frac{1}{\cancel{2}} m v^2 \frac{\cancel{2} \cdot 6}{3 \cdot 7} = \frac{2}{7} m v^2$$



$$\Delta E_p = \underbrace{mg\ell}_{\text{bar moves up a distance } \ell} + \underbrace{mg\frac{4\ell}{3}}_{\text{projectile moves up } 2 \times \frac{2\ell}{3}}$$

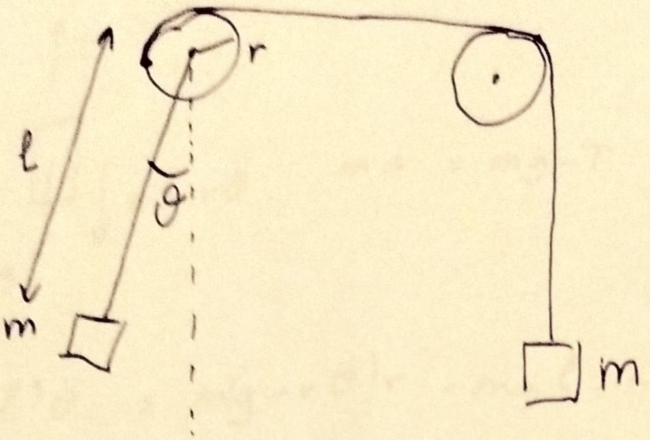
$$= mg\ell \cdot \frac{7}{3}$$

$$\frac{2}{7} \cancel{m} v^2 \geq \cancel{m} g \ell \frac{7}{3}$$

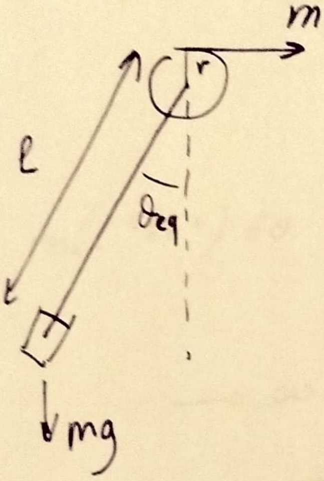
$$v^2 \geq \frac{49}{6} g \ell$$

$$v \geq 7\sqrt{\frac{g\ell}{6}}$$

Q 9



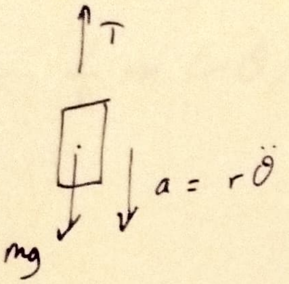
a) Find equilibrium angle θ_{eq}



$$mg\, l \sin\theta_{eq} = mg\, r$$

$$\sin\theta_{eq} = \frac{r}{l}$$

b) i) Tension in the cable is $-mr\ddot{\theta} + mg$



$$ma = mg - T$$

$$ml^2\ddot{\theta} = m(g - r\ddot{\theta})r - mgl\sin\theta$$

$$\theta = \theta_{eq} + \delta\theta$$

$$m(l^2+r^2)\delta\ddot{\theta} = mgr - mgl\sin(\theta_{eq}+\delta\theta)$$

$$= \cancel{mgr - mgl\sin\theta_{eq}}\underbrace{\cos\delta\theta}_{\approx 1} - mgl\cos\theta_{eq}\underbrace{\sin\delta\theta}_{\approx\delta\theta}$$

equilibrium

$$m(l^2+r^2)\delta\ddot{\theta} + mgl\cos\theta_{eq}\,\delta\theta = 0$$

$$\longrightarrow \boxed{\omega^2 = \frac{gl\cos\theta_{eq}}{r^2+l^2}}$$

b) ii) Using energy.

$$E = \frac{1}{2} m (r\dot{\theta})^2 + \frac{1}{2} m (l\dot{\theta})^2 + [-mgl(\cos\theta - 1)]$$
$$- mgr\theta$$

iii)

Equilibrium:

$$\frac{\partial E}{\partial \theta} = mgl\sin\theta - mgr = 0 \quad \rightarrow \sin\theta_{eq} = \frac{r}{l}$$

$$\theta = \theta_{eq} + \delta\theta$$

$$E = \frac{1}{2} m(r^2 + l^2)\delta\dot{\theta}^2 \overbrace{- mgr\theta_{eq}}^{\text{const.}} - mgr\delta\theta - mgl\cos(\theta_{eq} + \delta\theta) + \text{Const.}$$

$$\cos(\theta_{eq} + \delta\theta) = \underbrace{\cos\theta_{eq}}_{\text{const.}} \underbrace{\cos\delta\theta}_{\approx 1 - \frac{\delta\theta^2}{2}} + \sin\theta_{eq}\sin\delta\theta$$

$$= \frac{1}{2} m(r^2 + l^2)\delta\dot{\theta}^2 \cancel{- mgr\delta\theta} + mgl\cos\theta_{eq}\frac{\delta\theta^2}{2} \cancel{+ mgl\sin\theta_{eq}\delta\theta}$$

at equilibrium

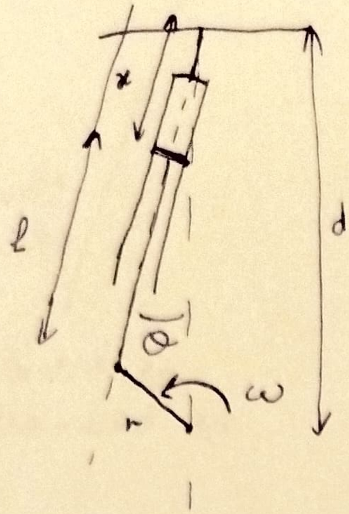
$$E = \frac{1}{2} m(r^2 + l^2)\delta\dot{\theta}^2 + mgl\cos\theta_{eq}\frac{\delta\theta^2}{2}$$

iv)

$$\frac{\partial E}{\partial t} = \left(m(r^2 + l^2)\delta\ddot{\theta} + mgl\cos\theta_{eq}\,\delta\theta\right)\delta\dot{\theta} = 0$$

$$\delta\ddot{\theta} + \omega^2\delta\theta = 0 \quad \text{with} \quad \boxed{\omega^2 = \frac{gl\cos\theta_{eq}}{r^2 + l^2}}$$

Q10 ~~long~~ Short



constant $\omega = \pi$ rad/s

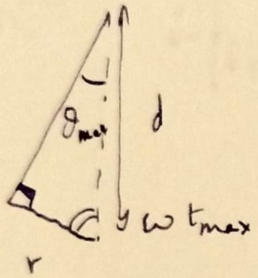
a) Sketch $x(t)$ and $\theta(t)$ in the interval $0 < t < 2$.

indicate the minimum and maximum value of θ and x, at well as the times at which they are reached

$r = \frac{d}{5}$ & $l = \frac{d}{2}$

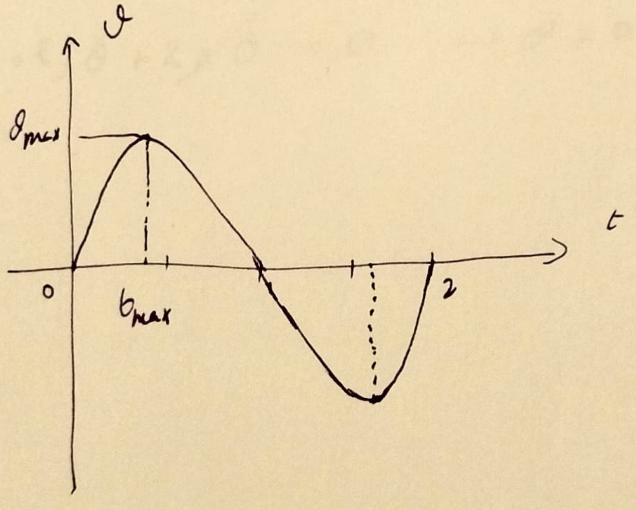
$x_{min} = d - r - l$ at $t = 0$ and $t = 2$ s

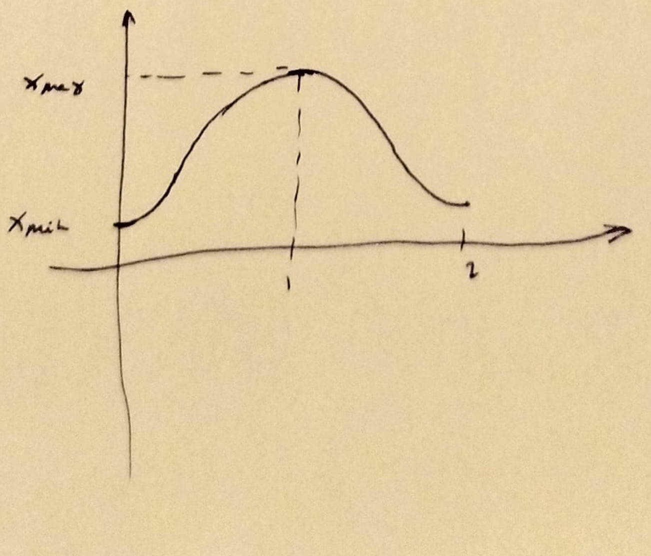
$x_{max} = d + r - l$ at $t = 1$ s



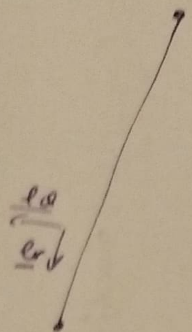
$\sin \theta_{max} = \frac{r}{d}$ $\theta_{max} \simeq 11.5°$

$\cos \omega t_{max} = \cancel{\frac{d}{r}} \frac{r}{d}$ $t_{max} \simeq 0.44$ s.



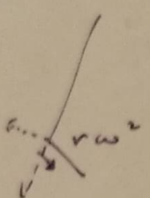


b)



$\ddot{x} - (x+l)\dot{\theta}^2 \quad \underline{e_r}$

$(x+l)\ddot{\theta} + 2\dot{x}\dot{\theta} \quad \underline{e_\theta}$



$$v = r\omega = (x+l)\dot{\theta} \longrightarrow \dot{\theta} = \frac{r\omega}{x+l} = \frac{r\omega}{d-r}$$

$$\dot{x} = 0$$

$$r\omega^2 = \ddot{x} - (x+l)\dot{\theta}^2 = \ddot{x} - (d-r)\dot{\theta}^2 = \ddot{x} - \frac{r^2\omega^2}{d-r}$$

$$\ddot{x} = r\omega^2 + \frac{r^2\omega^2}{d-r} = \left(1 + \frac{r}{d-r}\right) r\omega^2 = \frac{d}{d-r} \cdot r\omega^2$$

$$(x+l)\ddot{\theta} + \cancel{2\dot{x}\dot{\theta}} = 0 \longrightarrow \ddot{\theta} = 0$$

Solution to Part (a):

The moment of inertia of the flywheel is $J_1 = MR^2/2$.

The equation of motion of the disc-shaft system is essentially of a simple harmonic motion (SHM):

$$J_1\ddot{\alpha} + k\alpha = 0 \tag{1}$$

The angular natural frequency ω_n is therefore

$$\omega_\mathrm{n} = \sqrt{\frac{k}{J_1}} \tag{2}$$

Substituting the expressions for J_1 into (2) gives

$$\omega_\mathrm{n} = \sqrt{\frac{2k}{MR^2}} \tag{3}$$

(3) is the expression for the angular natural frequency of the system for torsional oscillations. Substituting M = 40 kg, R = 0.3 m, and k = 1200000 Nm/rad into (3) gives $\omega_\mathrm{n} = 816.50$ rad/s.

Solution to Part (b):

The general solution to the SHM described by (1) can be expressed as

$$\alpha = A\cos(\omega_\mathrm{n}t + \varphi) \tag{4}$$

Differentiating (4) with respect to time gives

$$\dot{\alpha} = -A\omega_\mathrm{n}\sin(\omega_\mathrm{n}t + \varphi) \tag{5}$$

Substituting initial conditions $\alpha = 0$ at $t = 0$ and $\dot{\alpha} = 200$ at $t = 0$ into (4) and (5) respectively gives

$$\begin{cases} \varphi = \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} \\ A = -\dfrac{200}{\omega_\mathrm{n}} \end{cases} \tag{6}$$

Substituting $\omega_\mathrm{n} = 816.50$ rad/s into the expression for A in (6) gives A = - 0.245 rad. Consequently, the amplitude of the motion is 0.245 rad. Note that an absolute value for A is used to describe the amplitude.

Solution to Part (c):

Note that the equation provided in the Data Sheet for estimating damping ratio ς is

$$\ln\left(\frac{y_1}{y_2}\right) = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}} \tag{7}$$

This equation relates the magnitudes of two successive peaks y_1 and y_2 to damping ratio ς. Students are expected to be able to work out the equation involving two magnitudes y_1 and y_{1+N}, where N is the number of oscillation cycles between the two magnitudes based on (7). The resultant equation can be expressed as

$$\ln\left(\frac{y_1}{y_{1+N}}\right) = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}}N \tag{8}$$

Substituting $y_{1+N} / y_1 = 1 - 95\% = 5\%$ and $N = 3$ into (8), gives

$$\ln\left(\frac{100}{5}\right) = \frac{2\pi\varsigma}{\sqrt{1-\varsigma^2}} \cdot 3 \tag{9}$$

Solving (9) for ς gives $\varsigma = 0.1568$.

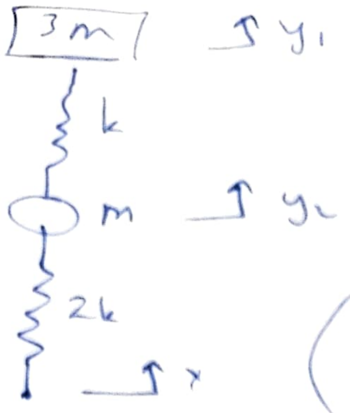
By definition, the damped nature frequency can be expressed as

$$\omega_{\mathrm{d}} = \omega_{\mathrm{n}}\sqrt{1-\varsigma^2} \tag{10}$$

Substituting $\varsigma = 0.1568$ into (10) gives

$$\frac{\omega_{\mathrm{d}}}{\omega_{\mathrm{n}}} = 98.76\% .$$

(a)



$$3m\ddot{y}_1 = k(y_2 - y_1)$$

$$m\ddot{y}_2 = k(y_1 - y_2) + 2k(x - y_2)$$

$$\begin{pmatrix} 3m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} = \begin{pmatrix} -k & k \\ k & -3k \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2kx \end{pmatrix}$$

$$M\ddot{\underline{y}} + K\underline{y} = \begin{pmatrix} 0 \\ 2k \end{pmatrix} x$$

$$M = \begin{pmatrix} 3m & 0 \\ 0 & m \end{pmatrix} \qquad K = \begin{pmatrix} k & -k \\ -k & 3k \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

To find the natural freq. and modes, we solve

$$\left([K] - \omega^2[M]\right)\underline{Y} = \underline{0} \quad , \text{assuming } \underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}\cos(\omega t)$$

To have non-zero $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $|K - \omega^2 M| = 0$

$$\begin{vmatrix} k-\omega^2 3m & -k \\ -k & 3k-\omega^2 m \end{vmatrix} = 0$$

$$(k-3\omega^2 m)(3k-\omega^2 m) - k^2 = 0$$

$$3k^2 + 3m^2\omega^4 - k\omega^2 m - 9k\omega^2 m - k^2 = 0$$

$$3m^2\omega^4 \quad -10km\omega^2 \quad 2k^2 = 0$$

$$\Delta = 100k^2m^2 - 4\cdot 2k^2\cdot 3m^2$$

$$= (100-24)k^2m^2 = 76\,k^2m^2$$

$$\omega^2 = \frac{10\,km \pm \sqrt{76}\,km}{6m^2} = \begin{cases} 0.21\ k/m \\ 3.12\ k/m \end{cases}$$

$$\begin{cases} \omega_1 = \pm 0.46\sqrt{\frac{k}{m}} \\ \omega_2 = \pm 1.77\sqrt{\frac{k}{m}} \end{cases}$$

Mode shapes:

$$\begin{pmatrix} k - 0.21 \cdot 3 \cdot \frac{k}{\cancel{m}} \cancel{m} & -k \\ -k & 3k - 0.21 \frac{k}{\cancel{m}} \cancel{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 - 3 \times 0.21 & -1 \\ -1 & 3 - 0.21 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

$$0.37\, y_1 = y_2 \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0.37 \end{pmatrix}$$

$$\begin{pmatrix} k - 3.12 \cdot 3\, k & -k \\ -k & 3k - 3.12\, k \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

$$-8.36\, y_1 = y_2 \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -8.37 \end{pmatrix}$$

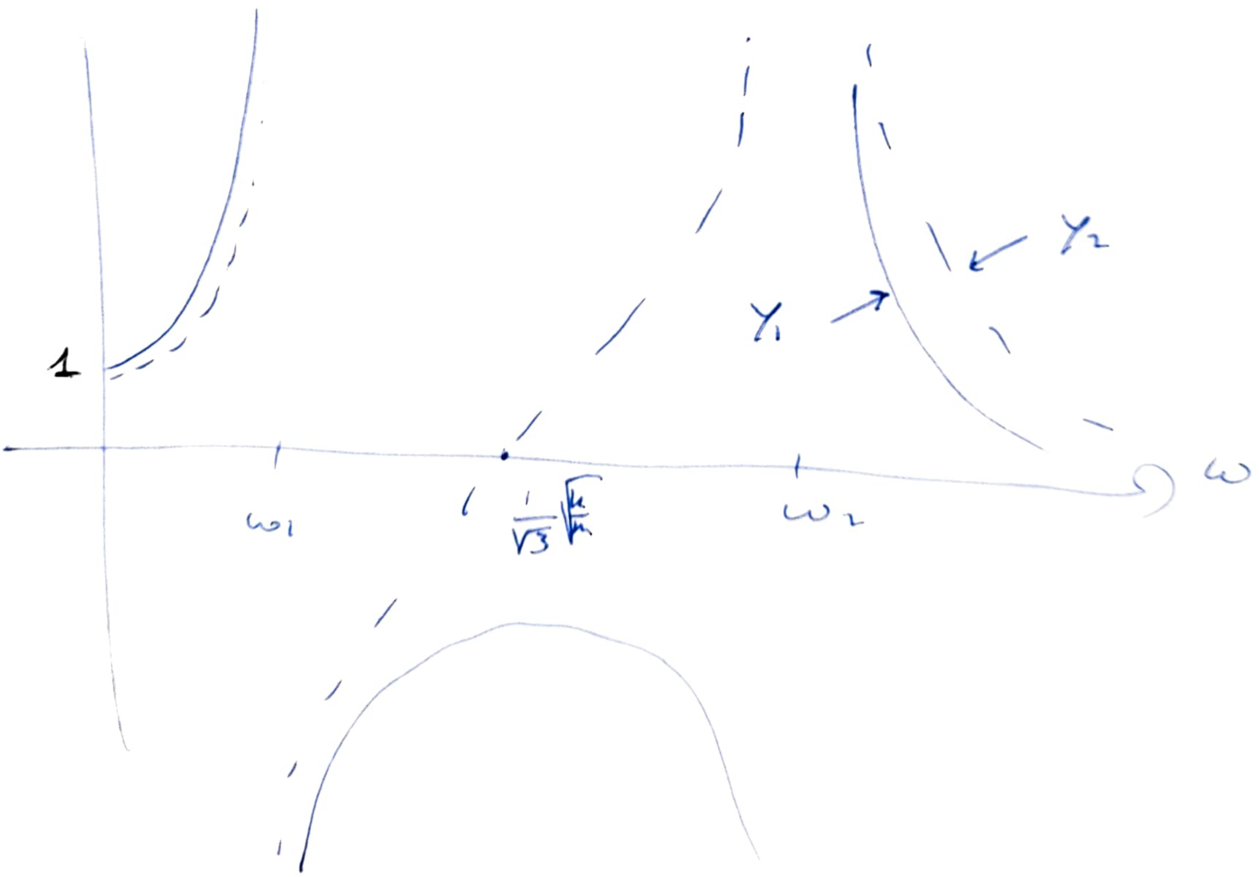
(b) For $x = X \sin(\omega t)$

$$(K - \omega^2 M)\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2kX \cancel{\sin\omega t} \end{pmatrix} \cancel{\sin\omega t}$$

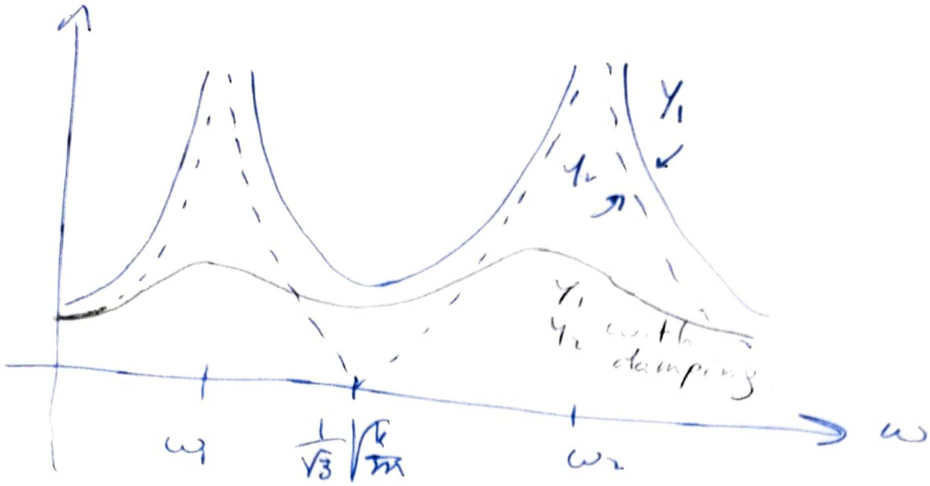
$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = (K - \omega^2 M)^{-1} \begin{pmatrix} 0 \\ 2kX \end{pmatrix}$$

$$= \frac{1}{\det(K - \omega^2 M)} \begin{pmatrix} 3k - \omega^2 m & k \\ k & k - \omega^2 3m \end{pmatrix} \begin{pmatrix} 0 \\ 2kX \end{pmatrix}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \frac{1}{3m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} \begin{pmatrix} 2k^2 X \\ (k - \omega^2 3m)\, 2kX \end{pmatrix}$$



(c)



Y_1
Y_2
Y_1
Y_2 with damping
ω_1
$\frac{1}{\sqrt{3}}\sqrt{\frac{k}{m}}$
ω_2
ω