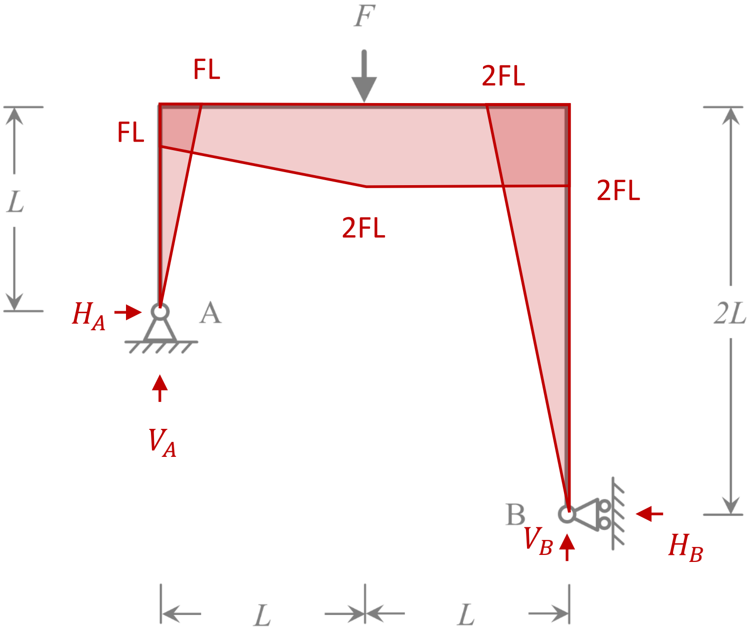




1) An arrangement that is somewhat counterintuitive but obeys all the usual rules for a statically determinate structure.

For global equilibrium…

$V_B = 0$

$V_A = F$

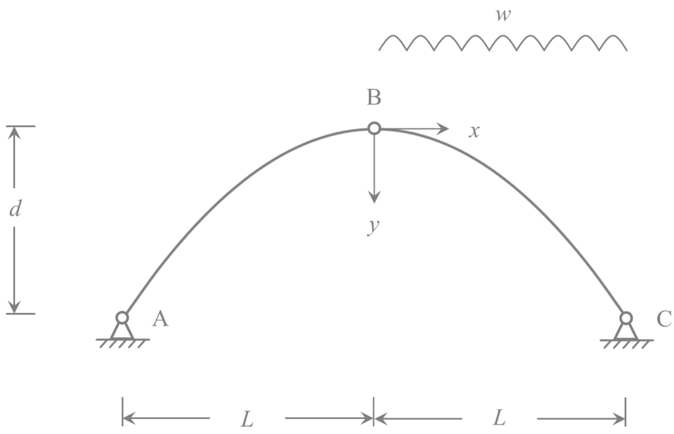
$H_A = H_B$

Taking moments about A: $H_B = -F$

Hence, $H_A = -F$

The minus sign simply indicates that the direction is opposite to the arrows initially drawn.

Point forces only, so expect linear bending moment diagram. Knowing boundary conditions (pin and vertical roller), and by equilibrium of free bodies, obtain the bending moment diagram shown. Bending moments drawn on the tension side.



2)

For global equilibrium…

$$H_A = H_C$$

$$V_A + V_c = wL$$

Taking moments about A:

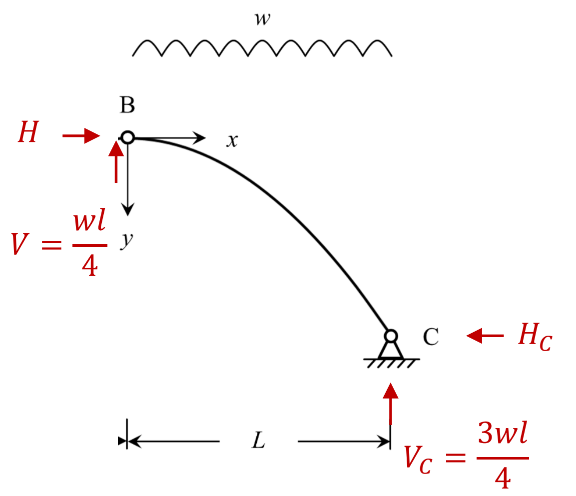
$$V_c = \frac{3wl}{4}$$

$$V_A = \frac{wl}{4}$$

Taking moments for RHS about B:

$$dH = \frac{3wL^2}{4} - \frac{wL^2}{2}$$

$$H = \frac{wL^2}{4d} = H_C = H_A$$



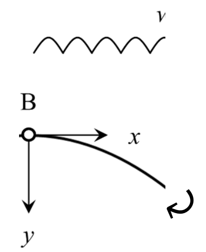
Or, since for AB reaction at A must pass through both pins:

$$H_C = H_A = V_A \frac{L}{d} = \frac{wL^2}{4d}$$

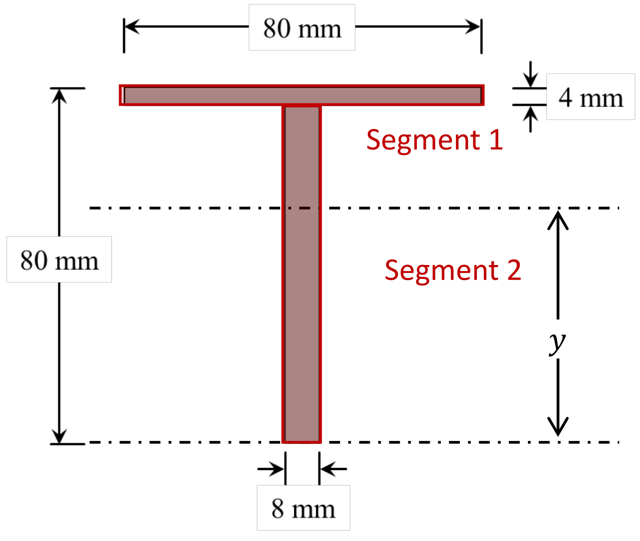
Taking moments about cut at x = L/2 on RHS

$$M_{x=\frac{l}{2}} = -\frac{wL^2}{4d}y - \frac{wlx}{4} + \frac{wx^2}{2} = -\frac{wL^2}{4d}y - \frac{wl^2}{8} + \frac{wl^2}{8} = -\frac{wL^2}{4d}y$$

Recalling that for this arch: $y = d\frac{x^2}{L^2}$



$$M_{x=\frac{L}{2}} = -\frac{wL^2}{4d} d\frac{x^2}{L^2} = -\frac{wL^2}{16}$$



$A_1 = 4 \times 80 = 320\, mm^2$

$A_2 = (2 \times 4) \times (80 - 4) = 608\, mm^2$

$A_{tot} = A_1 + A_2 = 320 + 608 = 928\, mm^2$

By first moments of area, measuring the lever arm y from the base...

$y_1 = 80 - \left(\frac{4}{2}\right) = 78\, mm$

$y_2 = \frac{(80-4)}{2} = 38\, mm$

$$y = \frac{A_1 y_1 + A_2 y_2}{A_{tot}}$$

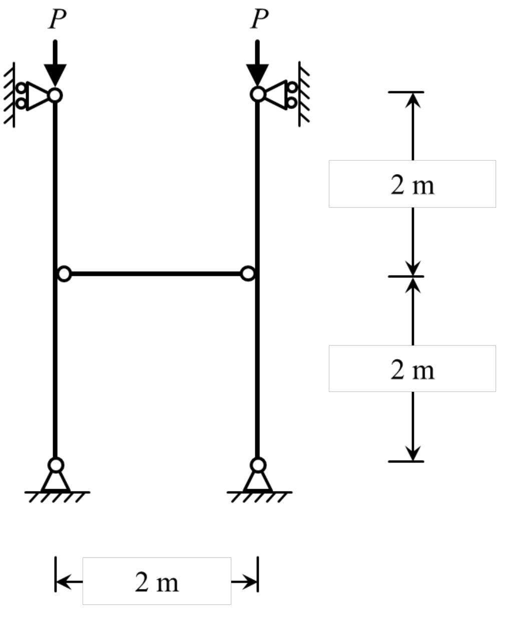
$$= \frac{320 \times 78 + 608 \times 38}{928} = \frac{48064}{928} = 51.8\, mm$$

$I_{tot} = I_1 + I_2 + A_1\overline{y_1}^2 + A_2\overline{y_2}^2$

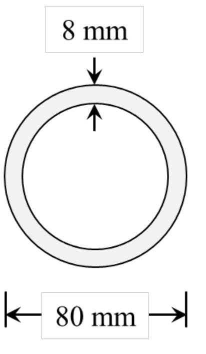
$$= \frac{80 \times 4^3}{12} + \frac{8 \times 76^3}{12} + 320 \times (80 - 2 - 51.8)^2 + 608 \times (51.8 - 38)^2$$

$= 427 + 292\,651 + 219\,661 + 115\,788$

$= 628\,527\, mm^4$



(a)



(b)

4) a)

$$I = \frac{\pi}{4}({r_o}^4 - {r_i}^4) = \frac{\pi}{4}(40^4 - 32^4) = 1\ 187\ 070\ mm^4$$

Note that the databook simplification for thin-walled sections ($I \approx \pi r^3 t = 1\ 172\ 593\ mm$) is sufficiently accurate here, provided the correct value of r is adopted, i.e., $r = r_o - \frac{t}{2} = 36\ mm$.

$$E = 15 \times 10^3\ Nmm^{-2}$$

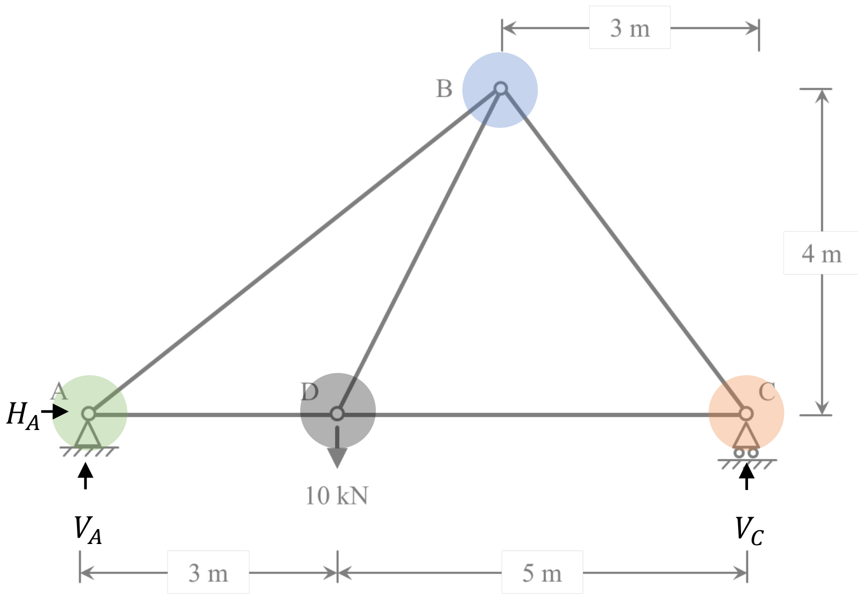
The horizontal member braces compression strut against compression strut at mid-height. At critical buckling load, neither can contribute to the buckling resistance of the other, i.e., this horizontal member provides no effective restraint against buckling. So, struts are pin ended with length 4.000 m…

$$L = 4000\ mm$$

Hence, Euler buckling load:

$$P_E = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 15\ 000 \times 1\ 187\ 070}{4\ 000^2} = 10\ 984\ N$$

b) The Euler buckling load assumes an imperfection free strut. A bamboo culm is a grown material which can be expected to exhibit a relatively high degree of imperfection compared to typical engineered structures.



5) a)

Probably quickest tackled graphically.

By polygon of forces, with applied load providing scale …

Reactions purely vertical.

$$V_A = \frac{5}{8} \times 10 = 6.25\ kN$$

$$V_C = \frac{3}{8} \times 10 = 3.75\ kN$$

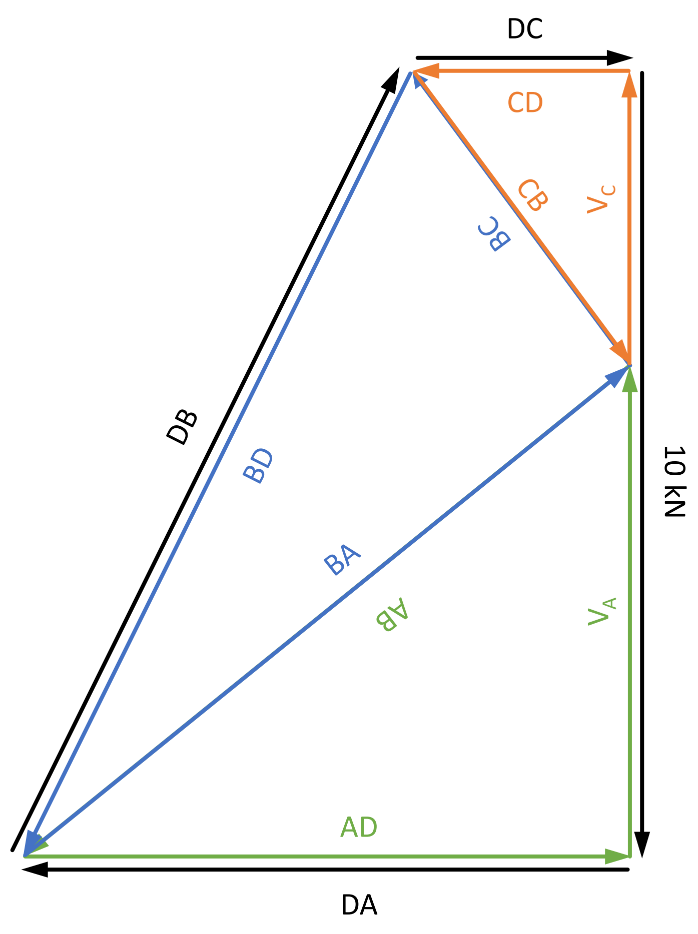
$$T_{AB} = -10.0\ kN$$

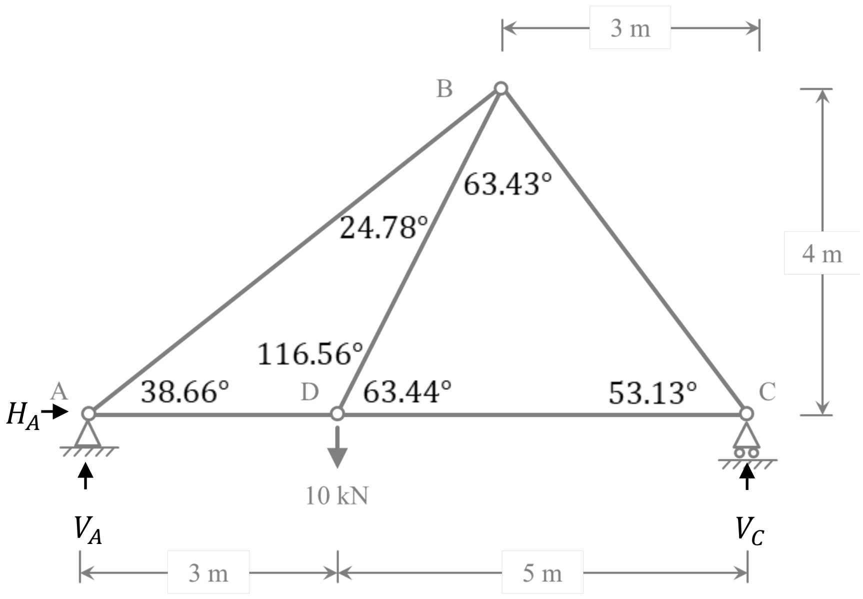
$$T_{BC} = -4.7\ kN$$

$$T_{AD} = 7.8\ kN$$

$$T_{BD} = 11.2\ kN$$

$$T_{CD} = 2.8\ kN$$





Alternatively, by calculation obtain the same result…

$$\angle BAD = \tan^{-1}\frac{4}{5} = 38.66°$$

$$\angle BCD = \tan^{-1}\frac{4}{3} = 53.13°$$

$$\angle CBD = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{4} = 63.43°$$

$$\angle ABD = \tan^{-1}\frac{5}{4} - \tan^{-1}\frac{2}{4} = 24.78°$$

$$\angle ADB = 180 - 38.66 - 24.78 = 116.56°$$

$$\angle CDB = 180 - 53.13 - 63.43 = 63.44°$$

Moments about A: $V_C = \frac{3\times 10}{8} = 3.75\ kN$

Vertical equilibrium: $V_A = 10 - 3.75 = 6.25\ kN$

Horizontal equilibrium: $H_A = H_C = 0$

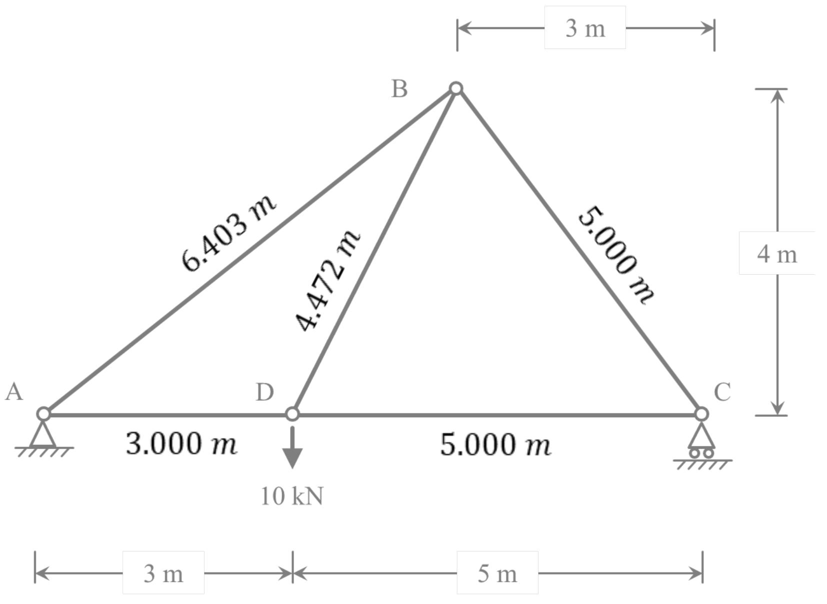
Resolving vertically at A: $T_{AB} = \frac{-6.25}{\cos(90-38.66)} = -10.0\ kN$

Resolving horizontally at A: $T_{AD} = 10 \times \cos 38.66 = 7.8\ kN$

Resolving vertically at C: $T_{BC} = \frac{-3.75}{\cos(90-53.13)} = -4.7\ kN$

Resolving horizontally at C: $T_{CD} = 4.69 \times \cos 53.13 = 2.8\ kN$

Resolving vertically at D: $T_{BD} = \frac{10}{\cos(90-63.44)} = 11.2 kN$



b)

Bar lengths can be measured from a scale drawing or determined by calculation, e.g.:

$AD = 3.000\ m$

$CD = 5.000\ m$

$BC = \sqrt{3^2 + 4^2} = 5.000\ m$

$AB = \sqrt{5^2 + 4^2} = 6.403\ m$

$\text{BD} = \sqrt{2^2 + 4^2} = 4.472\ m$

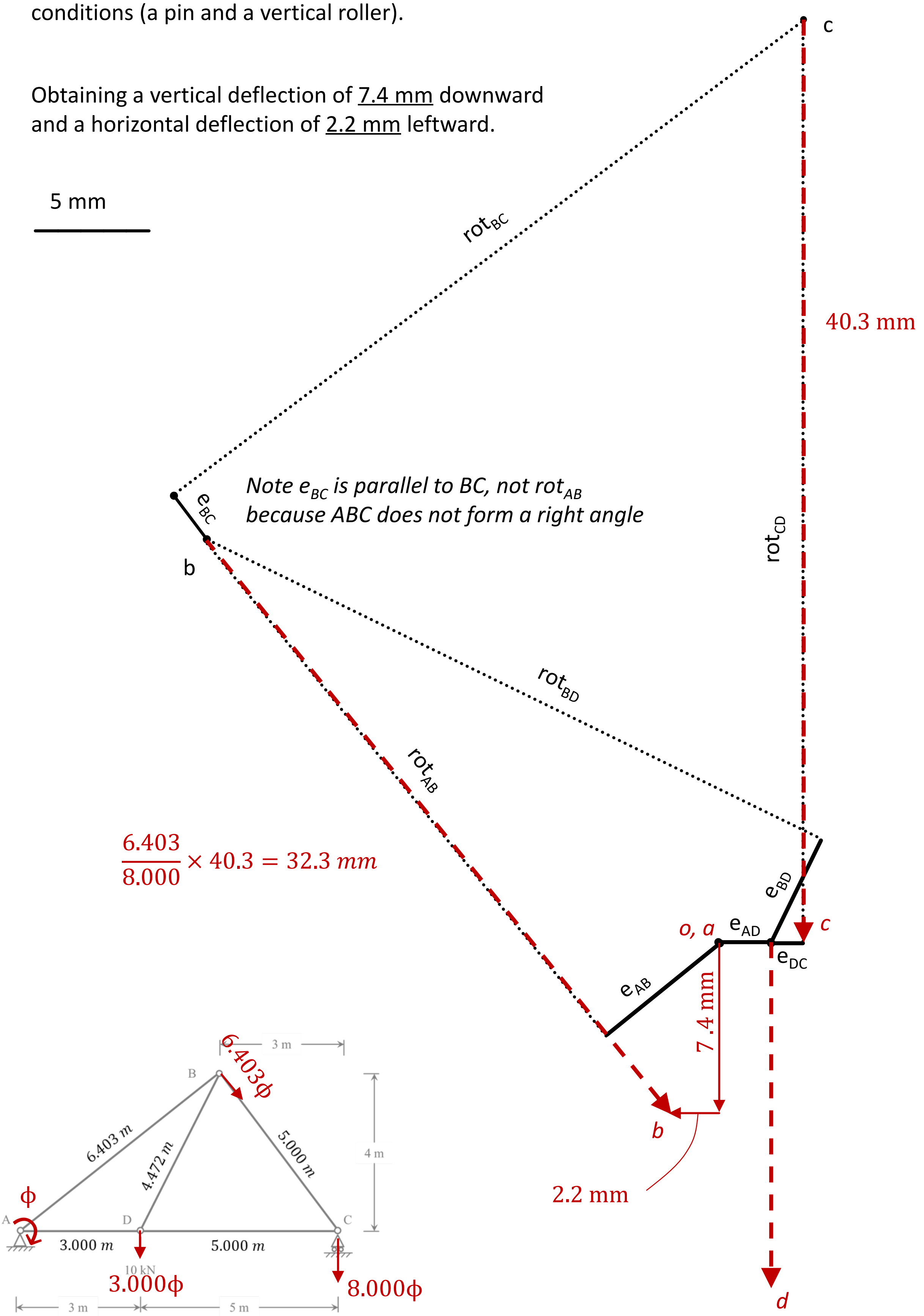
Using bar forces and length by either method, extensions may be determined knowing e=TL/EA where EA = 10^7 N as given in question.

Bar	T kN	Length mm	e=TL/EA mm
AD	7.8	3000	2.3
CD	2.8	5000	1.4
BC	-4.7	5000	-2.4
AB	-10.0	6403	-6.4
BD	11.2	4472	5.0

Solving by displacement diagram…

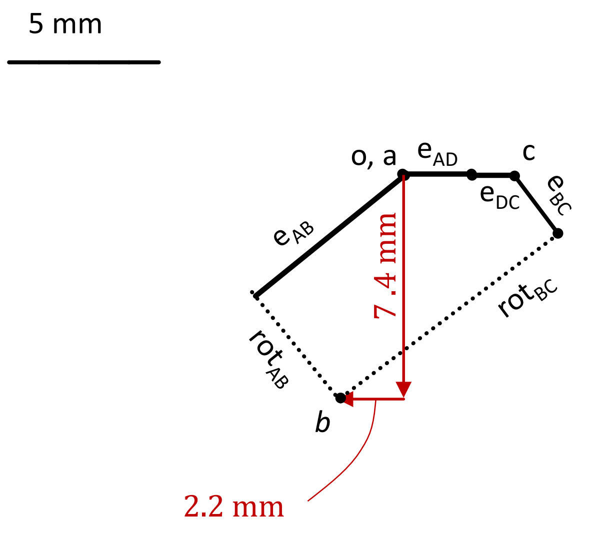
Draw displacement diagram, initially assuming (incorrectly) that AD remains horizontal. Then apply rigid body rotation about A, such that C is level with A in the final diagram, as required by the boundary conditions (a pin and a vertical roller).

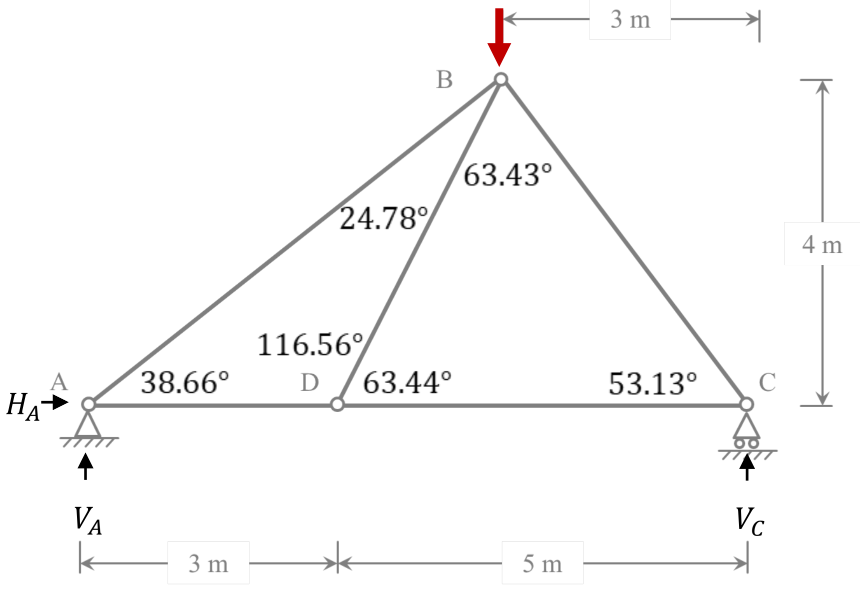
Obtaining a vertical deflection of <u>7.4 mm</u> downward and a horizontal deflection of <u>2.2 mm</u> leftward.



Or, remembering that this is an exam and we are only asked for the displacement of b … hence, JMA's much more efficient solution.

Obtaining a vertical deflection of <u>7.4 mm</u> downward and a horizontal deflection of <u>2.2 mm</u> leftward.





Displacements can also be found by virtual work.

$\sum F^* d = \sum T^* e$

We require real extensions and displacements so forces and tensions are virtual. Require vertical and horizontal components of displacement.

Virtual bar forces, vertical unit load downward at B:

Moments about A: $V_c = \frac{5\times1}{8} = 0.625\ kN$

Vertical equilibrium: $V_A = 10 - 6.25 = 0.375\ kN$

Horizontal equilibrium: $H_A = H_C = 0$

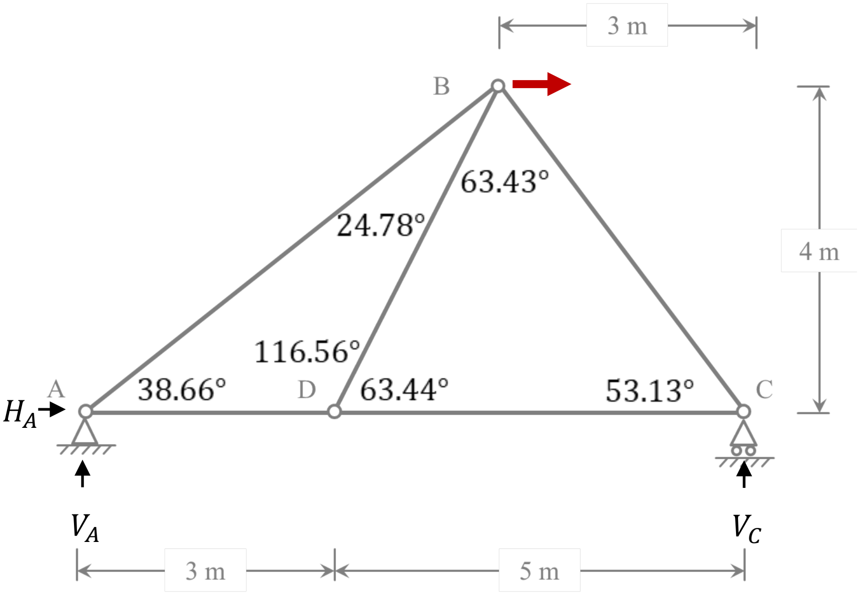
Resolving vertically at A: $T_{AB} = \frac{-0.375}{\cos(90-38.66)} = -0.600\ kN$

Resolving horizontally at A: $T_{AD} = 0.600 \times \cos 38.66 = 0.469\ kN$

Resolving vertically at C: $T_{BC} = \frac{-0.625}{\cos(90-53.13)} = -0.781\ kN$

Resolving horizontally at C: $T_{CD} = 0.781 \times \cos 53.13 = 0.469\ kN$

For equilibrium at D: $T_{BD} = 0\ kN$



Virtual bar forces, horizontal unit load rightward at B:

Moments about A: $V_c = \frac{4\times1}{8} = 0.500\ kN$

Vertical equilibrium: $V_A = -0.500\ kN$

Horizontal equilibrium: $H_A = -1.000\ kN$

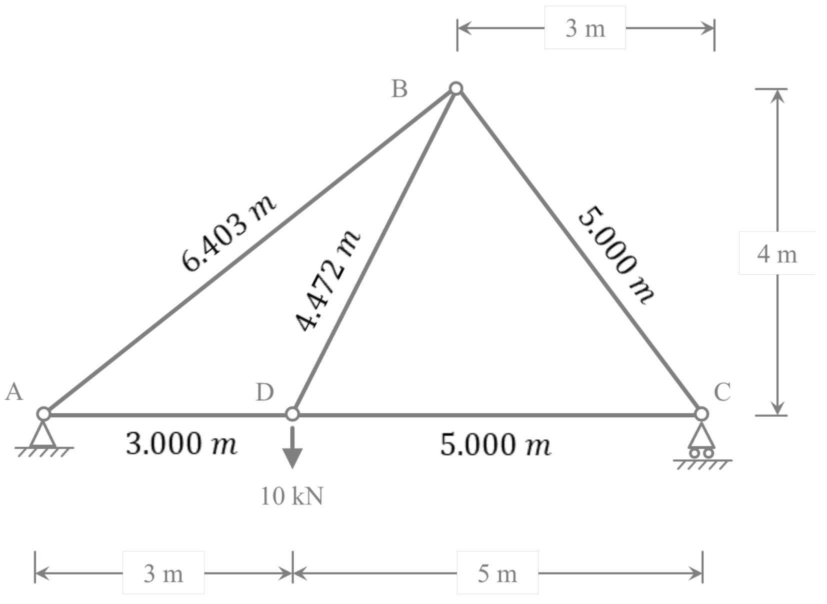
Resolving vertically at A: $T_{AB} = \frac{0.500}{\cos(90-38.66)} = 0.800\ kN$

Resolving horizontally at A: $T_{AD} = 1.000 - 0.800 \times \cos 38.66 = 0.375kN$

Resolving vertically at C: $T_{BC} = \frac{-0.500}{\cos(90-53.13)} = -0.625\ kN$

Resolving horizontally at C: $T_{CD} = 0.625 \times \cos 53.13 = 0.375\ kN$

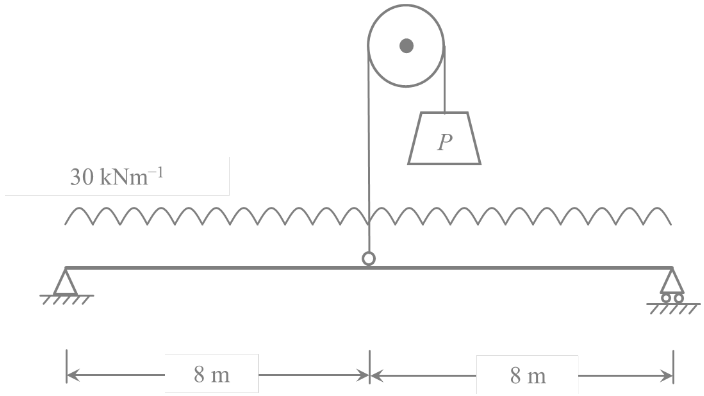
For equilibrium at D: $T_{BD} = 0\ kN$



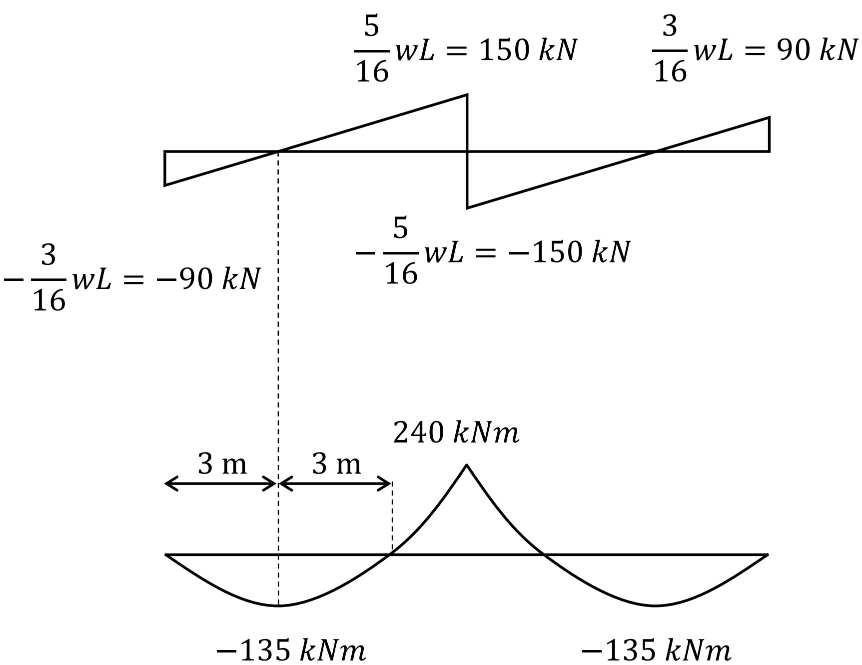
Giving a vertical displacement of <u>7.4 mm</u> downward and a horizontal displacement of <u>2.2 mm</u> leftward.

This method is entirely acceptable but probably slower than drawing a displacement diagram for those that are reasonably well practiced with graphical methods.

Bar	Force T kN	Length mm	e=TL/EA mm	T_v* kN	eT_v*	T_h* kN	eT_h*
AD	7.8	3000	2.3	0.469	1.079	0.375	0.863
CD	2.8	5000	1.4	0.469	0.657	0.375	0.525
BC	-4.7	5000	-2.4	-0.781	1.874	-0.625	1.500
AB	-10.0	6403	-6.4	-0.6	3.840	0.800	-5.120
BD	11.2	4472	5.0	0	0	0	0
Σ					<u>7.4</u>		<u>-2.2</u>



a)



b) Reducing P by 20% gives $P = \frac{4}{8}wL$ which tells us that the shear becomes $\frac{4}{16}wL = 120\ kN$ at each end and at the interior support. This gives zero moment mid-span and maximum bending moments as for two simply supported spans of length $\frac{L}{2}$:

$$\mathrm{M} = \frac{w\left(\frac{L}{2}\right)^2}{8} = \frac{30 \times 8^2}{8} = 240\ kNm$$

$$\mathrm{I} = \frac{bd^3}{12} = \frac{200 \times 600^3}{12} = 3.6 \times 10^9\ mm^4$$

$$\sigma = \frac{My}{I} = \frac{240 \times 10^6 \times \left(\frac{600}{2}\right)}{3.6 \times 10^9} = 20\ Nmm^{-2}$$

$20\ Nmm^{-2} < 24 Nmm^{-2}$ hence section is adequate.

c)

Deflection when P = 300kN noted as zero in the question.

When P = 240 kN…

From databook

$$\delta = \frac{5wL^4}{384EI} - \frac{PL^3}{48EI}$$

$$= \frac{5 \times 30 \times 16000^4}{384 \times 11 \times 10^3 \times 3.6 \times 10^9} - \frac{240 \times 10^3 \times 16000^3}{48 \times 11 \times 10^3 \times 3.6 \times 10^9} = 646 - 517 = 129\,mm$$

This is a deflection of approximately span / 124.

This represents a midspan displacement of less than 1 in 100, associated with a substantial change in the bending moment distribution in the beam. If we consider the analogous situation of an indeterminate beam with an intermediate support, rather than a cable attached to a weight, we can see that relatively small deviations in the vertical position of that support (due to settlement, low stiffness, tolerances, etc) can have drastic implications for the design of a beam for bending.

Any other pertinent, thoughtful comments would be acceptable.

SECTION B

1 (**short**)

(a) The curves represent:

A: ceramic - highest modulus, brittle failure

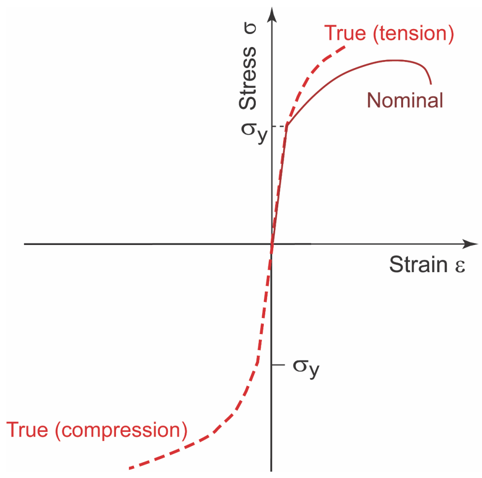
B: metal - modulus roughly half that of the ceramics, ductile response showing work hardening then necking before ductile fracture

C: polymer - low modulus and strength, brittle failure

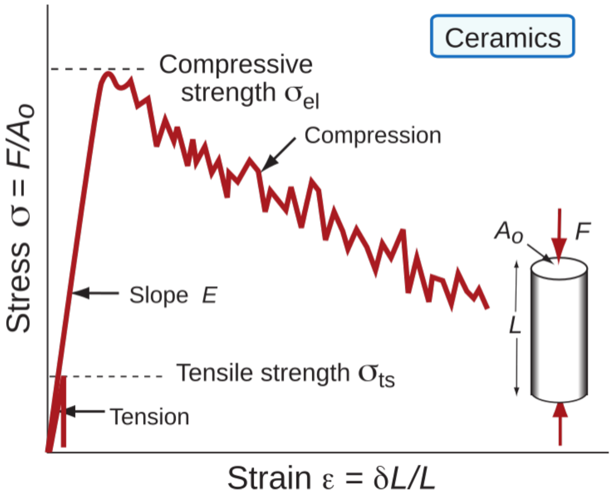
D: polymer - low modulus and strength, very high ductility (drawing)

(b) True stress > nominal stress, and true strain < nominal strain; hence true lies above and to the left of nominal in tension (terminating at the onset of necking).

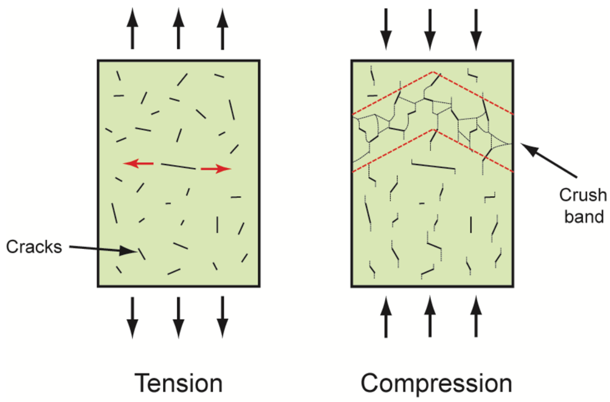
Compressive true stress-strain curve is identical in shape, but extends to larger strains (as necking is avoided).



(c) Ceramic in compression vs. tension:



Compressive strength $\gg$ tensile strength (roughly factor of 10)



The tensile strength is controlled by the growth of the "worst flaw (crack)". The compressive strength is controlled by "crushing", with cracks propagating stably until bands of material fail.

2 (**short**)

(a) Step strain response is a delta function for the stress, at the step location. Step stress response is a linear increase of strain over time.

(b) (i) d

(ii) e

(iii) g

(iv) f

3 (**short**)

(a) In the unit cell of sodium chloride, there are four Cl ions (as these form an FCC packing), and hence four Na ions (as the compound is NaCl). The mass of the unit cell is therefore

$$\frac{(4 \times 35.453 + 4 \times 22.989) \times 10^{-3}}{6.022 \times 10^{23}} = 38.819 \times 10^{-26}\,\mathrm{kg}$$

The volume of the NaCl unit cell is $\left(0.564 \times 10^{-9}\right)^3 = 17.94 \times 10^{-29}\,\mathrm{m}^3$. Hence the theoretical density

$$\frac{38.819 \times 10^{-26}}{17.94 \times 10^{-29}} \approx 2160\,\mathrm{kg/m}^3$$

(b) The length of the edge of the NaCl cell = diameter of Cl + diameter of Na. For a Cl diameter of unity, the unit cell size is thus 1.69, and the diagonal of one face = $\sqrt{2} \times 1.69 = 2.39$. Ions only touch along this diagonal if its length is equal to 2 (i.e. twice the Cl diameter). There are thus two equal gaps between Cl ions on the diagonal, equal to $0.39/2 = 0.195$ times the diameter of the Cl.

 (TURN OVER

4 (**short**)

(a) The plane strain condition indicates

$$\sigma_3 = \nu(\sigma_1 + \sigma_2). \tag{1}$$

Substituting (1) into 3D Hooke's law, we find

$$\epsilon_1 = \frac{1-\nu^2}{E}\sigma_1 - \frac{\nu(1+\nu)}{E}\sigma_2$$
$$\epsilon_2 = \frac{1-\nu^2}{E}\sigma_2 - \frac{\nu(1+\nu)}{E}\sigma_1.$$

Hence

$$C = \begin{bmatrix} \frac{1-\nu^2}{E} & -\frac{\nu(1+\nu)}{E} \\ -\frac{\nu(1+\nu)}{E} & \frac{1-\nu^2}{E} \end{bmatrix}$$

(b) For $\nu = 0.5$, we verify $\epsilon_1 + \epsilon_2 = 0$. Since $\epsilon_3 = 0$ in plane strain, $\sum_i \epsilon_i = 0$. This indicates that the material is incompressible.

5 (**long**)

(a) Describe the following concepts used in fracture mechanics:

(i) $K = Y\sigma\sqrt{\pi a}$ describes the loading at the tip of a sharp crack. It depends on the applied stress, and the specimen and crack geometry. It has dimensions stress $\times \sqrt{\text{length}}$.

(ii) K_{IC} is a material property, it depends on the material, not on geometry. It is the value of the stress intensity factor at a crack tip needed for a crack to propagate.

(iii) Fast fracture occurs when $K \geq K_{IC}$.

(b) $K_{IC} = Y\sigma_{\max}\sqrt{\pi a} \Rightarrow 28 \times 10^8 = Y \times (260 \times 10^6) \times \sqrt{\pi \times (1.2 \times 10^{-3})}$
$\therefore Y \approx 1.754$

$K = Y\sigma\sqrt{\pi a} = 1.754 \times (340 \times 10^6) \times \sqrt{\pi \times (0.6 \times 10^{-3})} = 25.89\,\text{MPa}\,\text{m}^{1/2}$
$\therefore K < K_{IC}$ thus no fracture.

(c) $\dfrac{da}{dN} = A\,\Delta K^n$, where a is the crack length, N is the number of fatigue cycles, ΔK is the stress intensity factor range, A and n are constants. It describes steady state crack propagation.

(i) $K_{IC} = 95\,\text{MPa}\,\text{m}^{1/2} = 95\,\text{MN}\,\text{m}^{-3/2}$
$\sigma_{\max} = 225\,\text{MPa}$ (tensile)
$\sigma_{\min} = 60\,\text{MPa}$ (compressive)
$a_0 = 2.5\,\text{mm}$
$A = 1.5 \times 10^{-10}\,\text{MN}^{-n}\,\text{m}^{1+1.5n}$
$n = 2.5$
$Y = 1$
Critical crack length
$K_{IC} = \sigma_{\max}\sqrt{\pi a_f} \Rightarrow 95 \times 10^6 = (225 \times 10^6) \times \sqrt{\pi a_f}$
$\therefore a_f = 56.7\,\text{mm}$

$$N_{\text{f}} = \int_{a_0}^{a_f} \frac{\mathrm{d}a}{A(Y\Delta\sigma\sqrt{\pi a})^n} = \frac{1}{A(Y\Delta\sigma\sqrt{\pi})^n}\int_{a_0}^{a_f} a^{-\frac{n}{2}}\,\mathrm{d}a = \frac{1}{A(Y\Delta\sigma\sqrt{\pi})^n}\left[\frac{a^{\left(1-\frac{n}{2}\right)}}{1-\frac{n}{2}}\right]_{a_0}^{a_f}$$

$$\therefore N_{\text{f}} = \frac{a_f^{\left(1-\frac{n}{2}\right)} - a_0^{\left(1-\frac{n}{2}\right)}}{A\left(1-\frac{n}{2}\right)\left(Y\Delta\sigma\sqrt{\pi}\right)^n}$$

Hence for $\Delta\sigma = 285\,\text{MPa}$, the fatigue life of the girder is

$$N_\mathrm{f} = \frac{a_f^{\left(1-\frac{n}{2}\right)} - a_0^{\left(1-\frac{n}{2}\right)}}{A\left(1-\frac{n}{2}\right)\left(\Delta\sigma\sqrt{\pi}\right)^n} = \frac{0.0567^{\left(1-\frac{2.5}{2}\right)} - 0.0025^{\left(1-\frac{2.5}{2}\right)}}{1.5\times10^{-10}\left(1-\frac{2.5}{2}\right)\left(285\sqrt{\pi}\right)^{2.5}} \approx 1.127\times10^4 \text{ cycles}$$

(ii) $N_{\mathrm{f}_1} = 5\times1.127\times10^4 = \dfrac{0.0567^{-0.25} - 0.0025^{-0.25}}{1.5\times10^{-10}(-0.25)\left(\Delta\sigma_1\sqrt{\pi}\right)^{2.5}}$

$$\Delta\sigma_1^{2.5} = \frac{0.0567^{-0.25} - 0.0025^{-0.25}}{1.5\times10^{-10}\times(-0.25)\times56350\times\pi^{1.25}}$$

$\therefore\ \Delta\sigma_1 \approx 150\,\mathrm{MPa}$

The decrease in stress range is $285 - 150 = 135\,\mathrm{MPa}$.

(iii) Use Miner's rule to estimate cumulative damage.

6 (**long**)

(a) (i) The shape factor for stiffness in bending of a beam section is defined as $\phi_e = \frac{I}{I_{\text{ref}}}$, where both the beam and the reference have the same cross-sectional area. For a solid square of side b, $A = b^2$ and $I_{\text{ref}} = \frac{b^4}{12} = \frac{A^2}{12}$, hence $\phi_e = \frac{12I}{A^2}$. The shape factor for strength in bending is defined as $\phi_f = \frac{I/y}{(I/y)_{\text{ref}}}$, with $(I/y)_{\text{ref}} = \frac{b^4}{12}\frac{2}{b} = \frac{A^{3/2}}{6}$. Hence, $\phi_f = \frac{6(I/y)}{A^{3/2}}$.

(ii) We wish to minimise the mass $m = \rho AL$. For the stiffness-limited design, the functional constraint is $\delta = \frac{5wL^4}{384EI} \leq \delta_{\text{max}}$. Substituting $I = \frac{A^2\phi_e}{12}$ (from the previous question) and solving for the free variable $A = \left(\frac{60wL^4}{384\delta_{\text{max}}\phi_e E}\right)^{1/2}$, we find

$$m = \left(\frac{5wL^6}{32\delta_{\text{max}}}\right)^{1/2} \times \frac{1}{M_e} \quad \text{with} \quad M_e = \frac{(\phi_e E)^{1/2}}{\rho}$$

the material performance index for the stiffness-limited design. For the strength-limited design, the functional constraint is $\sigma_0 = \frac{wL^2y}{8I} \leq \sigma_f$. Substituting $I/y = \frac{A^{3/2}\phi_f}{6}$ (from the previous question) and solving for the free variable $A = \left(\frac{6wL^2}{8\phi_f\sigma_f}\right)^{2/3}$, we find

$$m = \left(\frac{9w^2L^7}{16}\right)^{1/3} \times \frac{1}{M_f} \quad \text{with} \quad M_f = \frac{(\phi_f\sigma_f)^{2/3}}{\rho}$$

the material performance index for the strength-limited design.

(iii) We obtain:

	stiffness-limited		strength-limited	
Materials	M_e	rank	M_f	rank
Steel	14.9	3	66.1	2
Al alloy	22.5	1	61.1	3
Ti alloy	16.8	2	86.7	1

(b) (i) The design requirements are:

- Objective: minimise the mass m
- Geometric constraints: L and R specified, $t \ll R$

- Functional constraints: maximum deflection (stiffness), does not fail (strength)
- Free variables: choice of material, tube thickness t

(ii) Using $I = \pi R^3 t$, the stiffness-limited deflection reads

$$\delta = \frac{5wL^4}{384EI} = \frac{5wL^4}{384\pi ER^3 t} \leq \delta_{\max} \quad \Leftrightarrow \quad t \geq \frac{5wL^4}{384\pi ER^3\delta_{\max}} = t_e\,,$$

and the strength-limited deflection

$$\sigma_0 = \frac{M_0 y}{I} = \frac{wL^2 R}{8\pi R^3 t} \leq \sigma_f \quad \Leftrightarrow \quad t \geq \frac{wL^2}{8\pi R^2 \sigma_f} = t_f\,.$$

We obtain:

Materials	t_e (mm)	t_f (mm)	limiting constraint
Steel	**0.99**	0.11	stiffness
Al alloy	**2.96**	0.50	stiffness
Ti alloy	**2.07**	0.14	stiffness

The value of t for the active constraint (the largest value for each material) is in bold.

(iii) The only relevant shape factor is ϕ_e. Using the result of (a.i) with the expressions $A = 2\pi Rt$ and $I = \pi R^3 t$, we find

$$\phi_e^{\text{tube}} = \frac{12I}{A^2} = \frac{12\pi R^3 t}{4\pi^2 R^2 t^2} = \frac{3R}{\pi t}\,.$$

With this expression, the shape factor and mass $m = \rho AL$ are:

Materials	ϕ_e^{tube}	m (kg)	rank
Steel	19.4	0.97	1
Al alloy	6.5	0.97	1
Ti alloy	9.2	1.17	2

We will need to consider other constraints (cost, environmental resistance) to decide between steel and Al alloy.

END OF PAPER