RMF/BL/6

1P2 2023 Crib Section A

1)

Statically determinate, so solvable purely by equilibrium.

$h_C = \ h_A = 0$

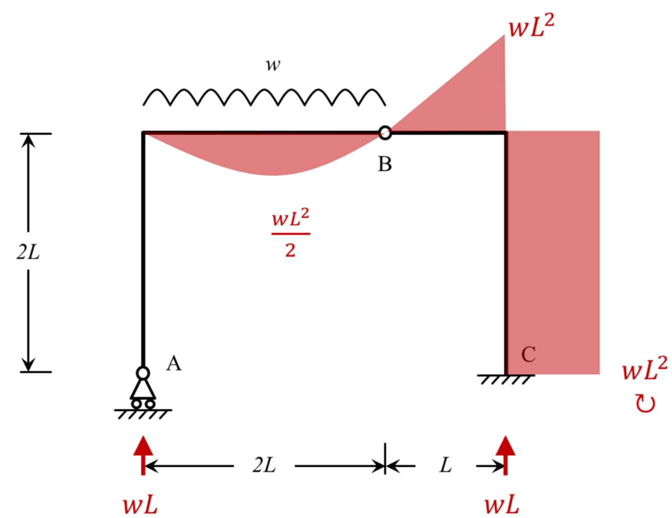
$v_A = \dfrac{2wL^2}{2L} = wL$

$v_C = 2wL - v_{\mathrm{A}} = wL$

Taking moments about B: $v_c L = wL^2 = M_c$ [clockwise]

Moment at midspan under load: $v_a L - \frac{wL^2}{2} = wl^2 - \frac{wl^2}{2} = \frac{wl^2}{2}$ [counter clockwise]

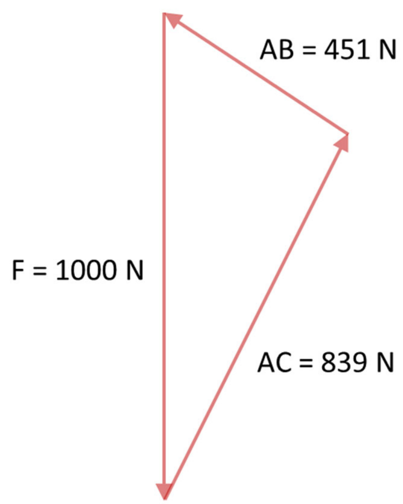
Taking further free bodies as needed finds:



2)

Determine bar forces…

By force polygon:



Or by calculation:

$$v_C = \frac{3F}{4}$$

$$v_B = \frac{F}{4}$$

$$h_C = \frac{3F}{8}$$

$$h_b = \frac{3F}{8}$$

$$T_{AC} = \left(\frac{3\sqrt{5}}{8}\right)F = 839\ N$$

$$T_{AB} = \left(\frac{\sqrt{13}}{8}\right)F = 451\ N$$

Determine bar lengths by scaling or calculation:

$$L_{AC} = \sqrt{5}L$$

$$L_{AB} = \sqrt{13}L$$

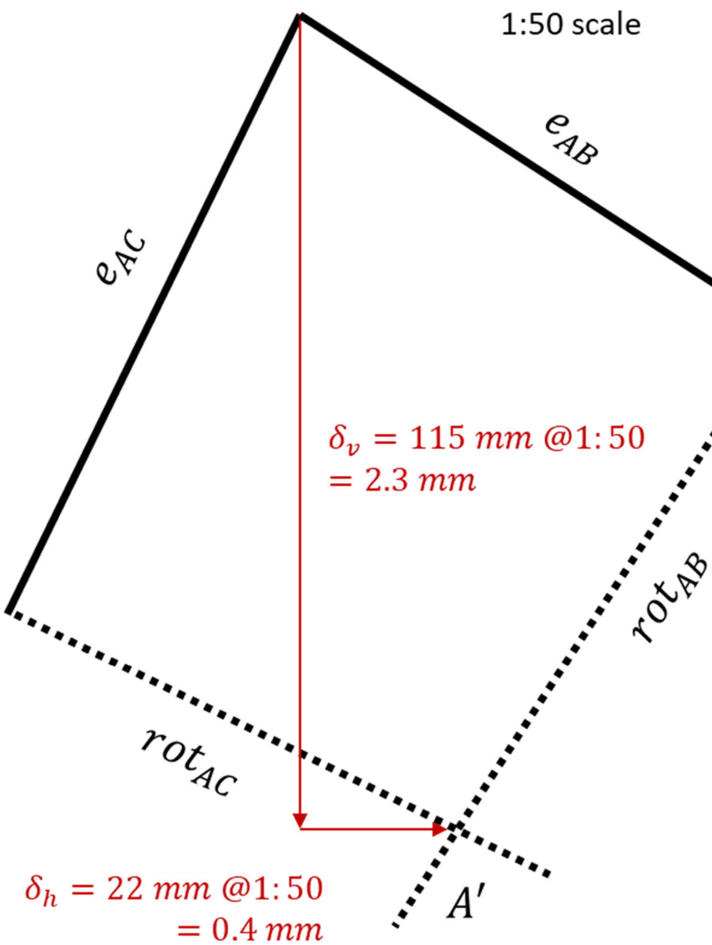
Determine extensions:

$$e_{AC} = \frac{TL}{AE} = \frac{15}{8} = 1.875\ mm$$

$$e_{AB} = \frac{TL}{AE} = \frac{13}{8} = 1.625\ mm$$

Find displacements…

graphically:



Or by virtual work:

$$\sum T^* e = \sum F^* \delta$$

unit force down gives:

$$\delta_v = 0.839 \times 1.875 + 0.451 \times 1.625 \;=\; 2.3\,mm$$

unit force right gives:

$$\delta_h = -0.559 \times 1.875 + 0.901 \times 1.625 \;=\; 0.4\,mm$$

3)

Cable, so built in support condition still acts as pin.

Solve using equilibrium and known dip at midspan.

Vertical reactions:

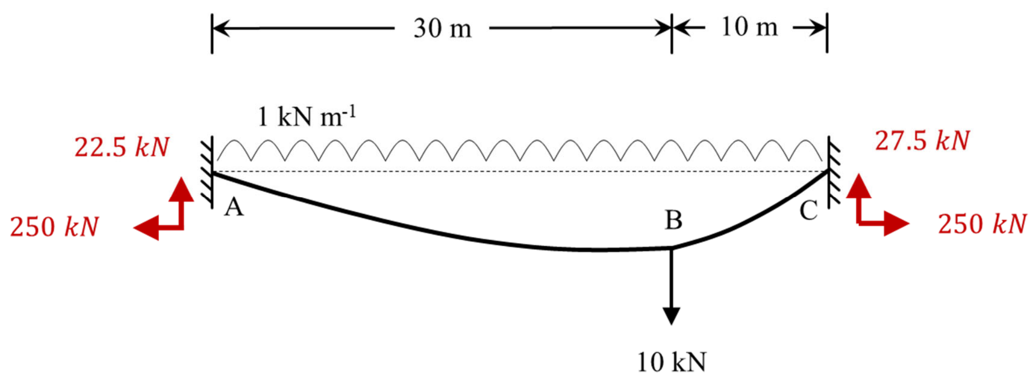
$v_A + v_C = 40 + 10 = 50\ kN$

Taking moments about A: $v_C = \frac{(40\times20+30\times10)}{40} = 27.5\ kN$

Therefore: $v_A = 22.5\ kN$

Horizontal reactions:

Taking moments about B: $h_A = \frac{20v_A - 1\times20\times10}{1} = 250\ kN = h_C$



Finding dip d…

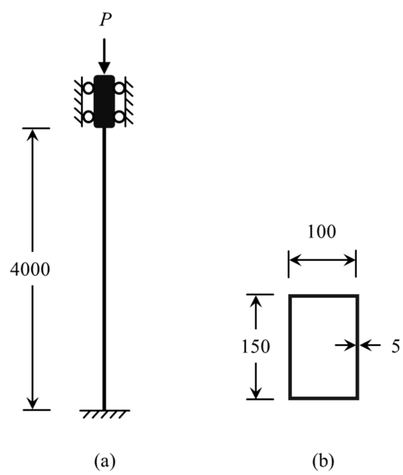
Taking moments at B: $v_C * 10 - 1 \times 10 \times 5 - h_C \times d = 0$

Therefore: $d = 900\ mm$

4)

$\sigma_y = 275\ MPa$

$E = 210\ GPa$



Note buckling will occur about weak axis…

$$I_{minor} = \frac{150 \times 100^3}{12} - \frac{140 \times 90^3}{12} = 3995000\ mm^4$$

$$A = 150 \times 100 - 140 \times 90 = 2400\ mm^2$$

Fixed-fixed end conditions so: $L' = 0.5L = 2000\ mm$

Euler $P_{crit} = \frac{\pi^2 EI}{L'^2} = \frac{\pi^2 \times 210 \times 10^3 \times 4 \times 10^6}{2000^2} = 2072\ kN$

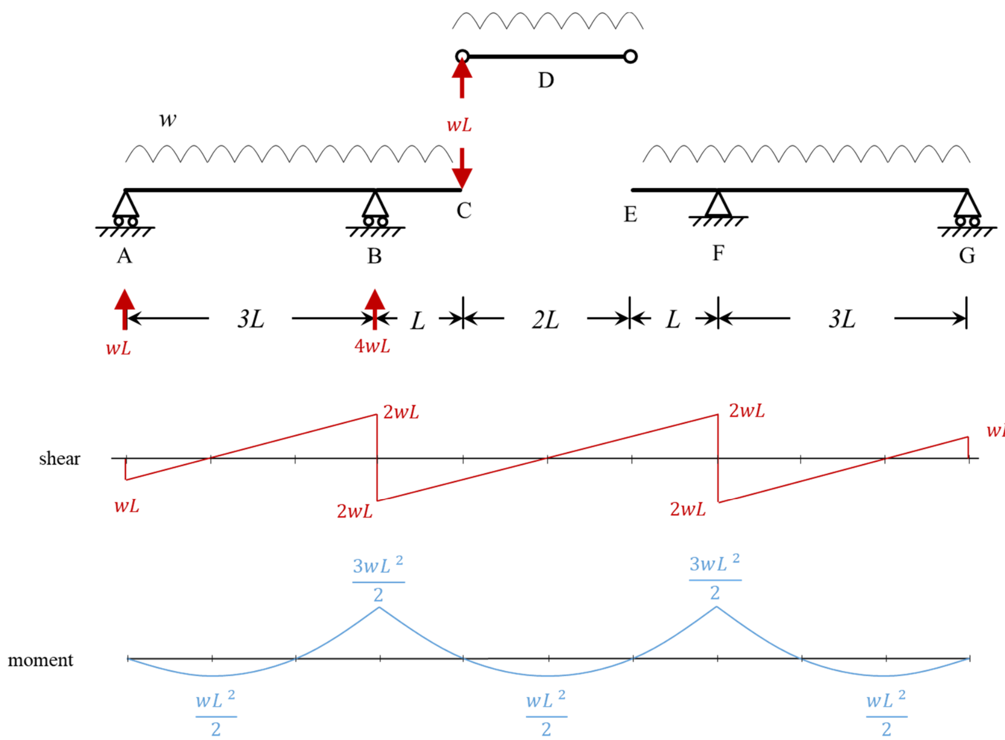
Crushing: $\sigma_y A = 275 \times 2400 = 660\ kN$

So crushing at $P = 660\ kN$ governs.

5)

a)

Statically determinate, symmetrical 'drop-in' span arrangement. Equations of equilibrium and free bodies obtain:



b)

Superposition of databook cases…

Require deflection at C + deflection of simply supported beam CE at D.

Deflection of simply supported beam CE at D: $\delta_1 = \frac{5w(2L)^4}{384EI} = \frac{80wL^4}{384EI}$

For deflection at C require deflection of cantilever BC and deflection due to rotation at base of cantilever at B.

Deflection of cantilever BC is superposition of deflection due to distributed load w and point load wL at C: $\delta_2 = \frac{wL^4}{8EI} + \frac{(wL)L^3}{3EI} = \frac{11wL^4}{24EI}$

Moment at base of cantilever and hence at B is $M = \frac{3wL^2}{2}$

Deflection due to rotation at base of cantilever at B: $\delta_3 = \left[\frac{3wL^2}{2}\frac{3L}{3EI} - \frac{w(3L)^3}{24EI}\right]L = \left[\frac{36wL^3}{24EI} - \frac{27wL^3}{24EI}\right]L = \frac{9wL^4}{24EI}$

So total deflection $\delta_{total} = \frac{80wL^4}{384EI} + \frac{11wL^4}{24EI} + \frac{9wL^4}{24EI} = \frac{80wL^4}{384EI} + \frac{176wL^4}{384EI} + \frac{144wL^4}{384EI}$

$$= \frac{400wL^4}{384EI} = \frac{25wL^4}{24EI}$$

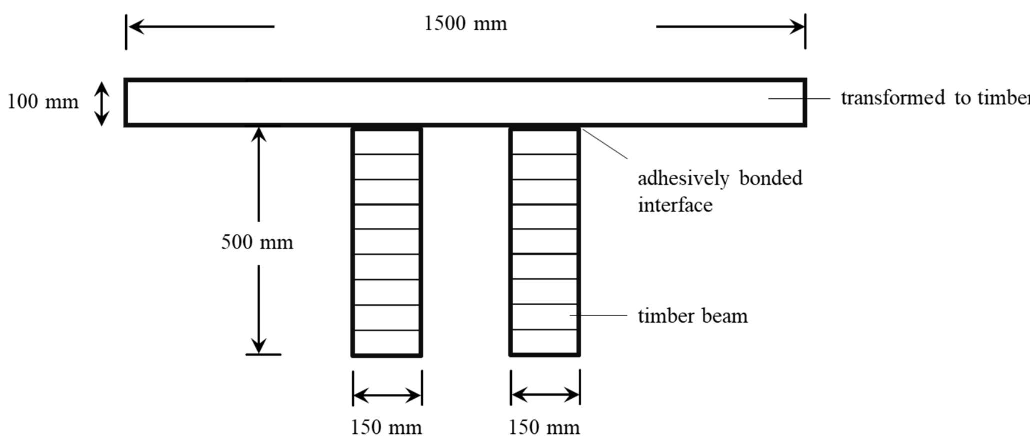
6)

a)

Transform section, e.g. to timber...

$$\frac{E_{concrete}}{E_{timber}} = \frac{30\ GPa}{10\ GPa} = 3$$

Determine transformed second moment of area with flange width multiplied by 3 …



Find neutral axis by first moment of area, taken from base:

$$\bar{y} = \frac{100 \times 1500 \times 550 + 300 \times 500 \times 250}{100 \times 1500 + 300 \times 500} = 400\ mm$$

b)

Find second moment of area by parallel axis theorem:

$$I = \frac{1500 \times 100^3}{12} + 1500 \times 100 \times (150)^2 + \frac{300 \times 500^3}{12} + 300 \times 500 \times (150)^2$$

$$= 125 \times 10^6 + 3375 \times 10^6 + 3125 \times 10^6 + 3375 \times 10^6 = 10 \times 10^9\ mm^4$$

Hence:

$$EI = 10 \times 10^3 \times 10 \times 10^9 = 100 \times 10^{12}\ N\ mm^2$$

Note that transformation to concrete leads to an E that is three times greater and an I that is one-third as great...

so EI is the same!

c)

Need to check top and bottom fibres. $M = EI\kappa$

$$k = \frac{\sigma_{max}}{y_{max}E}$$

$$M = (EI)\ \frac{\sigma_{timber}}{y_{bottom}E_{timber}} = 100 \times 10^{12}\ mm^4 \times \frac{20}{400 \times 10 \times 10^9} = 500 \times 10^6\ Nmm$$

$$M = (EI)\ \frac{\sigma_{concrete}}{y_{top}E_{concrete}} = 100 \times 10^{12}\ mm^4 \times \frac{30}{200 \times 30 \times 10^9} = 500 \times 10^6\ Nmm$$

So balanced failure and $\boldsymbol{M_{max} = 500\ kN\ m}$

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Question7

(a)

Vickers hardness is a measure of the strength of the material. It can be calculated by measuring the indent size produced under load by a diamond pyramid. The projected area of indent $A = \frac{d^2}{2} = 0.1152\text{mm}^2$. The yield stress σ_y is

$$\sigma_y \approx \frac{H}{3} = \frac{1}{3}\frac{Fg}{A} = \frac{1}{3} \times \frac{5 \times 9.8}{0.1152} \approx 141.8 \text{ MPa}. \tag{1}$$

(b)

Substitute $\sigma_t = \sigma_y = 141.8$ and $\epsilon_t = 0.08$, K is calculated as

$$K = \frac{141.8}{\sqrt{0.08}} \approx 500 \text{ MPa}. \tag{2}$$

From the definition of the true strain

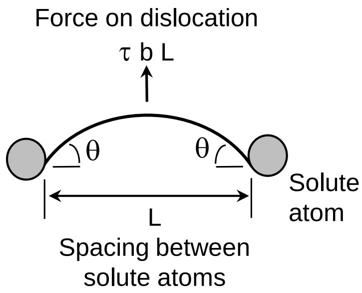
$$\epsilon_t = \ln(1 + \frac{\Delta L}{L_0}) \tag{3}$$

The true strain at $\sigma_t = 200\text{MPa}$ is $(200/500)^2 = 0.16$. Therefore, the change of length can be calculated from

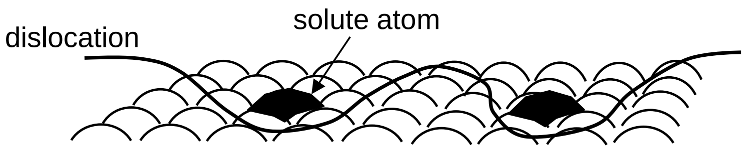
$$\Delta L = L_0(e^{\epsilon_t} - 1) = 20 \times (e^{0.16} - 1) = 3.47 \text{ cm} \tag{4}$$

Question 8

(a) Substitutional solid solution atoms are different in size to the host atoms, “roughening”



the slip plane. Dislocations are pinned where they meet these atoms, bowing out until the force on the dislocation τ b L is sufficient to pull the dislocation past the solute atom – the bowing angle θ is generally much less than 90^o, and the hardening is modest.



(b) (i) σ_o is the intrinsic yield strength of the pure element, the minimum stress needed to move a dislocation through the lattice due to the imperfect bonding around the dislocation core.

(ii) From the Cu end of the graph:

σ_o = 60 MPa for pure Cu

σ_y = 110 MPa for 10 wt% Cu, and σ_y = 130 MPa for 20 wt% Cu

Hence: $110 = 60 + \alpha\, 10^n$ and $130 = 60 + \alpha\, 20^n$

Taking logs and solving for *n*: $n = 0.485$ (≈ 0.5)

Substituting back to find α: α = 16.4 MPa (with n = 0.485), or α = 15.8 MPa (with n = 0.5) .

(iii) From the Ni end of the graph:

σ_o = 80 MPa for pure Ni

σ_y = 140 MPa for 10 wt% Cu (i.e. 90 wt% Ni)

So the solid solution hardening *increment* for 10wt% solute is 50 MPa for Ni in Cu, and 60MPa for Cu in Ni – so the hardening factor α is higher for Cu in Ni (both being multiplied by 10^n).

Question 9

(a) The stress amplitude σ_a = 1200/2 = 600 MPa. Reading directly from the graph, $N_f \approx 2\times10^6$ cycles.

(b) Use Goodman's rule to convert from the stress range ($\Delta\sigma_{\sigma m}$) with mean stress σ_m to an equivalent stress range ($\Delta\sigma_{\sigma o}$) for zero mean stress:

$$\Delta\sigma_{\sigma_o} = \frac{\Delta\sigma_{\sigma m}}{\left(1 - \frac{\sigma_m}{\sigma_{ts}}\right)} = \frac{1200}{\left(1 - \frac{100}{1100}\right)} = 1320 \text{ MPa.}$$

Hence the amplitude for zero mean is σ_a = 1320/2 = 660 MPa.

From the *S* – *N* curve, the lifetime is about 3×10^5 cycles. The percentage reduction in lifetime is about 85%, i.e. a reduction of lifetime by a factor of around 7 – a large effect.

(c) The fatigue limit is the stress amplitude at which the S – N curves level off giving an infinite life for amplitudes below this level. We can either change the design to reduce the stresses (as in mountain bikes, which have larger tubes with a larger second moment of area to reduce the stresses), or change the material to one with a higher fatigue limit.

(d) The change in section will cause a stress concentration, increasing the stress by a factor depending on the change in section size and the internal corner radius at this location. The stress concentration factor can be reduced by increasing the corner radius, bringing the magnified stress below the fatigue limit.

Question10

(a)

Young's modulus of glass is governed directly by atomic packing and atomic bonding. Young's modulus of polyethylene is governed by molecular packing and bonding.

It is easier to modify the young's modulus of polyethylene as one can change the molecular weight and/or degree of crystallinity via processing. There is little scope for manipulating the Young's modulus of glass as it is micro-structure insensitive.

(b)

(i)

Bending of the solid ligaments of the foam is the dominating mechanism for its elastic response.

(ii)

The density of the foam can be computed from the scaling law, as

$$\rho_f = \sqrt{\frac{E_f}{E_s}}\rho_s = \sqrt{\frac{1.3}{200}} \times 8.9 = 0.72\ Mg\ m^{-3} \tag{5}$$

Therefore, the density of the foam is 0.72 $Mg\ m^{-3}$

(iii)

The Nickel foam has a density of 0.72 $Mg\ m^{-3}$ and a Young's modulus of 1.3 GPa. According to Fig. 3.1 in the Materials data book, this is comparable to the typical wood which has densities of 0.6 $\sim$ 1 $Mg\ m^{-3}$ and Young's moduli of 0.5 $\sim$ 3 GPa. The properties that determine the suitability could be cost, thermal/electrical conductivity or embodied energy.

Question 11

(a) Shape factor $\phi_e = 12I/A^2$

Objective: embodied energy $H = m\,H_m = \rho\,AL\,H_m$

Constraint: bending stiffness $S = C_1EI/L^3 = C_1\,E\phi_e\,A^2/12\,L^3$

Eliminate *A* in objective equation using constraint: $A = (12\,S\,L^3/C_1\,E\phi_e)^{1/2}$

Hence $H = (12\,S\,L^5/C_1)^{1/2} \times \frac{\rho\,H_m}{(E\phi_e)^{1/2}}$

S, *L*, C_1 are constant, so performance index to maximise = $\frac{(E\phi_e)^{1/2}}{\rho\,H_m}$

	1000 × index	Rank
Steel	259.6	2
Al alloy	65.8	4
GFRP	73.4	3
Softwood	960.4	1

(b) (i) Diameter *D* = 60mm, so radius *R* = 30mm, thickness *t* = 2.5mm

Using exact formula for I (Structures databook): $I = (\pi/4)\,(R^4 - (R-t)^4)$

$(E\,I)_{steel} = 210\ \times (\pi/4)\,(3^4 - (2.75)^4) = 3926.8$ GPa.cm^4

(This-walled approximation, $I = \pi\,R^3\,t$, gives *I* = 21.21 cm^4, instead of 18.70 cm^4, so not sufficiently accurate).

For a solid section of radius *R* = 30mm: $I = \pi\,R^4/4 = 63.63$ cm^4

$(E\,I)_{Al} = 4580$ GPa.cm^4 (✓ 17% stiffer than steel)

$(E\,I)_{GFRP} = 1272$ GPa.cm^4 (✗ not stiff enough)

$(E\,I)_{wood} = 591$ GPa.cm^4 (✗ 17% not stiff enough)

(ii) Area of steel = $\pi(3^2 - 2.75^2) = 4.516$ cm^2

and mass of steel/unit length, $(\rho\,A)_{Steel}$ = $7800 \times (4.516 \times 10^{-4})$ = 3.523 kg/m

Area of Al = $\pi \times 3^2 = 28.27$ cm^2

and mass of Al/unit length, $(\rho\,A)_{Al}$ = $2800 \times (28.27 \times 10^{-4})$ = 7.917 kg/m

Embodied energy/unit length = $\rho\,A\,H_m$

Embodied energy/unit length of steel = 3.523×32 = 112.7 MJ/m

Embodied energy/unit length of Al = 7.917×190 = 1504 MJ/m

(iii) Frame transport energy: 500 km, 1.5 MJ/tonne.km

Steel has lower mass per unit length (and therefore lower transport energy per unit length), but also has the lower embodied energy/unit length – so for a fixed length of frame it will have the lower combined energy.

Mass of 40m of steel = 40 × 3.523 = 140.92 kg

Transport energy = 1.5 × 0.14092 × 500 = 105.7 MJ

Embodied energy = 140.92 × 32 = 4509 MJ

Frame total: 4615 MJ

(c) Embodied energy in panels:

	Mass/panel (kg)	H_m (MJ/kg)	H (8 panels) (MJ)
Si	14.4	120	13824
Glass	1.8	39	562
Al	1.8	190	2736
		TOTAL:	17122 MJ

Panel transport energy: 11,500 km, 0.18 MJ/tonne.km, mass = 8 × 18 = 144kg

Transport energy (sea) = 0.18 × 0.144 × 11,500 = 298 MJ

Transport energy (land) = 1.5 × 0.144 × 700 = 151 MJ

Panel total: 17571 MJ

Total energy (frame and panel, embodied and transport) = 22186 MJ

(d) Average power = 300W

In one year, energy output = 300×60×60 = 2628 kWh/year = 9461 MJ/year

(i) Energy payback time = 22186 / 9461 = 2.34 years

(ii) Usual cost of 2628 kWh/year = 2628 × 0.40 = £1051 per year

Cost payback time = 7000 / 1051 = 6.66 years

Question12

(a)

The hoop stress σ_h and axial stress σ_t of the pressure vessel is given by

$$\sigma_h = \frac{Pr}{t}, \qquad \sigma_t = \frac{Pr}{2t}. \tag{6}$$

Since $\sigma_h > \sigma_t$, the critical flaw size a_{crit} is determined by the hoop stress. At fast fractrure,

$$K = K_{IC} = Y\sigma_h\sqrt{\pi a_{\text{crit}}} \tag{7}$$

Therefore,

$$a_{\text{crit}} = \frac{1}{\pi Y^2}(\frac{K_{IC}}{P}\frac{t}{r})^2. \tag{8}$$

If $a_{\text{crit}} > t$, gas will leak out through the crack before the crack is large enough to propagate under fast fracture conditions.

(b)

From the Weibull statistics:

$$P_s(V) = \exp(-\frac{V}{V_0}(\frac{\sigma}{\sigma_0})^m) \tag{9}$$

Let L_1, σ_1 denote the length and failure stress of the rod with length 1 m, and L_2, σ_2 denote those of the rod with length 0.4 m. From the equation above

$$\exp(-\frac{V_1}{V_0}(\frac{\sigma_1}{\sigma_0})^m) = \exp(-\frac{V_2}{V_0}(\frac{\sigma_2}{\sigma_0})^m). \tag{10}$$

Therefore

$$m = \frac{\ln(V_1/V_2)}{\ln(\sigma_2/\sigma(1))} = \frac{\ln(L_1/L_2)}{\ln(\sigma_2/\sigma(1))} \approx 8.7. \tag{11}$$

The failure stress of a 0.2m long rod is thus

$$\sigma_f = (\frac{L_1}{L_f})^{1/m}\sigma_1 = 1.083 \text{ GPa} \tag{12}$$

(c)

Let P_{sb} denote the survival probability of the turbine blade, and let $P_{s1} = 0.99$ denote the survival probability of the Nickel rod given in (b) with $\sigma_1 = 0.9$ and $V_1 = AL_1 = 0.25\text{m}^3$. From the Weibull formula

$$P_{sb} = \exp\big(-\int_a^b (\frac{\sigma(x)}{\sigma_0})^m \frac{A_b}{V_0} dx\big) \tag{13}$$

$$P_{s1} = \exp\big(-(\frac{\sigma_1}{\sigma_0})^m \frac{V_1}{V_0}\big). \tag{14}$$

Therefore,

$$
\begin{aligned}
\frac{\ln P_{sb}}{\ln P_{s1}} &= \frac{A_b}{\sigma_1^m V_1} \int_a^b \sigma(x)^m dx && (15)\\
&= \frac{A_b}{\sigma_1^m V_1} \int_a^b (-6x + 2.4)^m dx && (16)\\
&= \frac{0.1}{0.9^{8.7} \times 0.25} \Big(-\frac{1}{6(m+1)} (-6x + 2.4)^{m+1} |_{0.2}^{0.4} \Big) && (17)\\
&\approx 0.1, && (18)
\end{aligned}
$$

with $m = 8.7$ obtained from Part (b). The survival probability of the blade is therefore

$$P_{sb} = P_{s1}^{0.1} = 0.99^{0.1} = 0.999. \tag{19}$$

And the failure probability of the blade is $1 - P_{sb} = 0.001 = 0.1\%$.