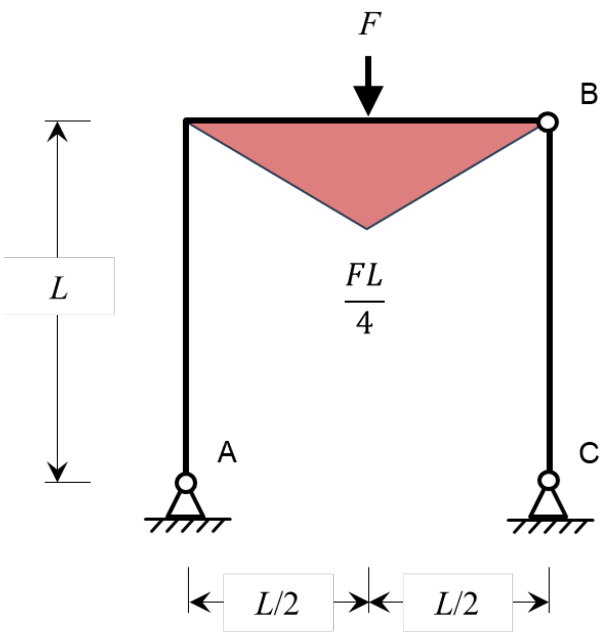
Q1 Short

The structure is a three-pin portal, hence statically determinate. No horizontal reaction is possible at C as BC is pinned both ends. So, by equilibrium:

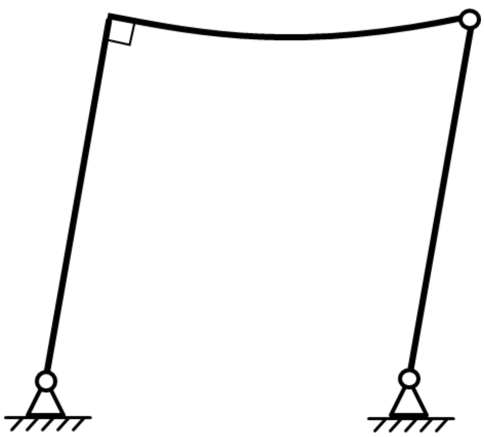
$$H_C = H_A = 0$$

$$V_A = V_C = \frac{F}{2}$$

$$M_{max} = \frac{F}{2} \times \frac{L}{2} = \frac{FL}{4}$$



If zero moment at the fixed corner seems counterintuitive, consider the deflected shape...



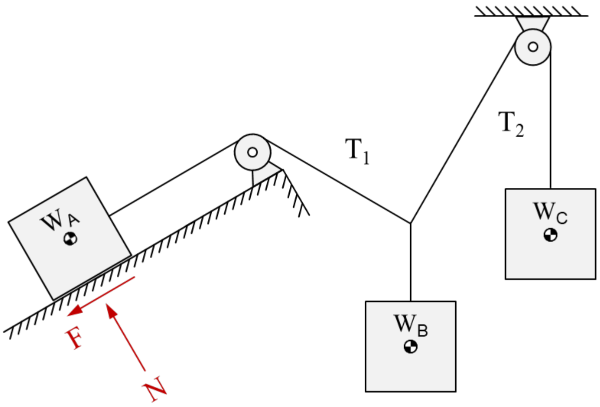
Examiner's comments...

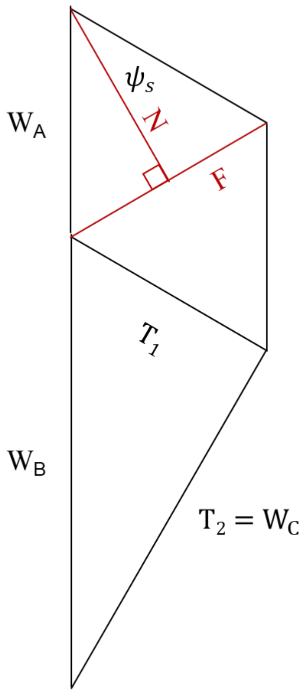
"A statically determinate 3-pin frame subject to a point load. Done well by most. A significant number drew an appropriately shaped bending moment diagram but calculated the wrong maximum value of moment. Some candidates did not recognise that the horizontal reactions must be zero and hence there must be no bending moments in the columns of the frame – meaning that the 'beam' acts as simply supported. Some candidates drew shear force diagrams even though these were not requested."

Q2 Short

The static friction angle between WA and the slope is equal to the angle of the slope, so there is no maximum weight for the block as it will never slide down the slope. We thus need to determine the minimum weight of WA required and the weight of WB.

The system is statically determinate, so by equilibrium, we can quickly solve graphically:





Or by calculation:

1) $W_B = \frac{1}{2}T_1 + \frac{\sqrt{3}}{2}T_2$

2) $\frac{\sqrt{3}}{2}T_1 = \frac{1}{2}T_2 \Rightarrow T_1 = \frac{1}{\sqrt{3}}T_2$

Subbing 2) into 1): $W_B = \frac{1}{2}\frac{1}{\sqrt{3}}T_2 + \frac{\sqrt{3}}{2}T_2 \Longrightarrow T_2 = \frac{\sqrt{3}}{2}W_B$

Since the pulley is frictionless:

$T_2 = W_C = \frac{\sqrt{3}}{2}W_B$, hence $T_1 = \frac{1}{2}W_B$

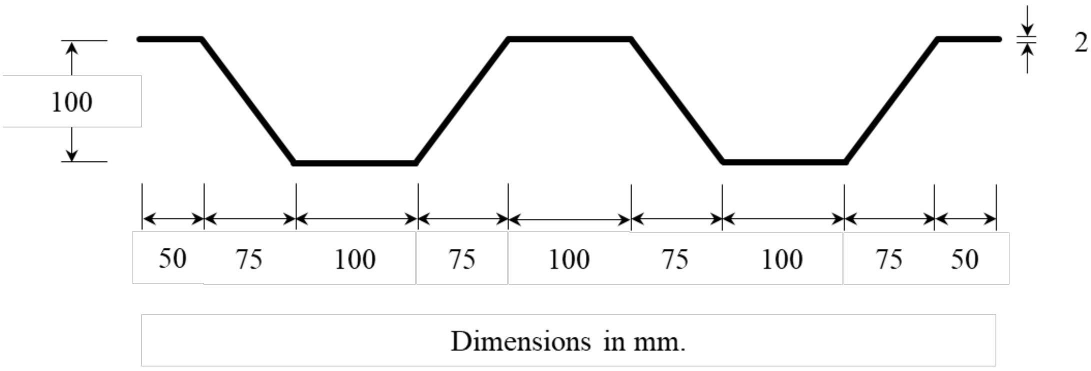
The coefficient of friction is the tangent of the friction angle: $\mu_s = \tan\psi_s = \tan 30 = \frac{\sqrt{3}}{3}$, the normal force N is $\frac{\sqrt{3}}{2}W_A$,and the component of the weight force parallel to the slope is $\frac{1}{2}W_A$.

So: $\frac{1}{2}W_A + \frac{\sqrt{3}}{3}\frac{\sqrt{3}}{2}W_A = T_1 = \frac{1}{\sqrt{3}}T_2 = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}W_B \Longrightarrow W_A \geq \frac{1}{2}W_B$

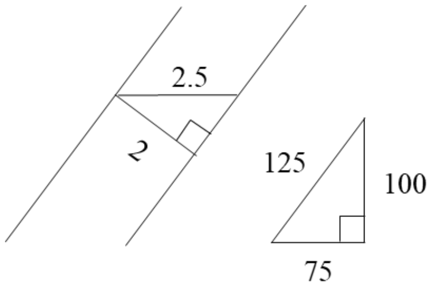
Examiner's comments…

"A system of blocks in equilibrium, either suspended or in contact with slope. Many candidates correctly calculated the weights of one or other of the two blocks requested, but fewer correctly calculated both. In general candidates who used graphical methods had more success, mostly because it helped them understand the problem rather than just following equilibrium calculations. Some candidates produced reams of undirected equilibrium-like calculations without really getting anywhere."

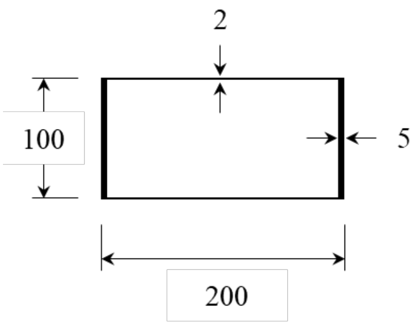
Q3 Short



Horizontal distribution of material does not affect the second moment of area about the horizontal neutral axis. So, the inclined webs can be converted to equivalent vertical webs of the same height but with a commensurately increased thickness of 2.5 mm.



The second moment area of the section is then most easily calculated as that of an equivalent box section:



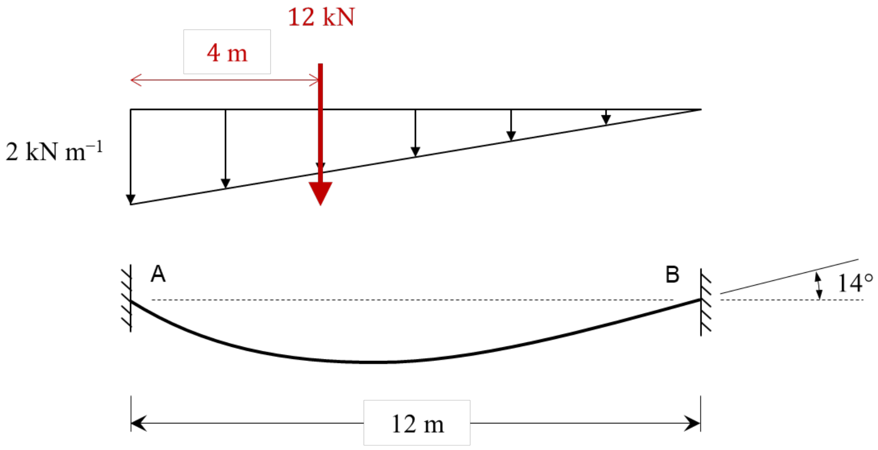
$$I = \frac{200 \times 100^3}{12} - \frac{190 \times 96^3}{12} = 2.7 \times 10^6 \; mm^4$$

One could also calculate from first principles or by parallel axis theorem, but this would almost certainly be slower and more error prone.

Examiner's comments…

"Second moment of area of a trapezoidal decking profile. Done well by most. A number of approaches and simplifications were in evidence. Some of these simplifications were excessive – for example a number of candidates treated the inclined segments as equivalent to a vertical segment of the same height, but without adjusting the equivalent thickness."

Q4 Short



Flexible cable so cannot sustain bending moments, statically determinate so solve by equilibrium. Taking moments about A:

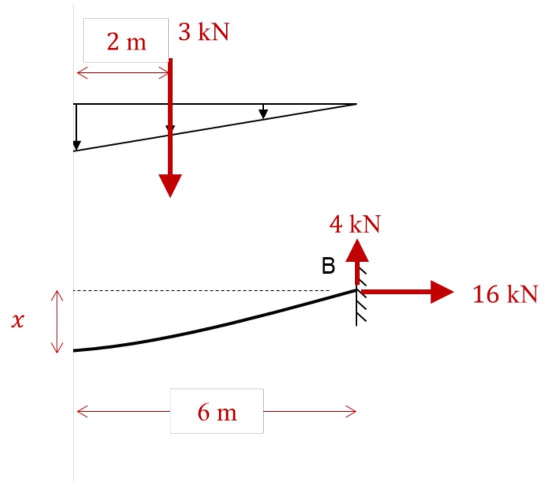
$$V_B = \frac{\left(\frac{2-0}{2} \times 12\right) \times \left(\frac{1}{3} \times 12\right)}{12} = 4\ kN$$

And by vertical equilibrium: $V_A = 8\ kN$

Since the cable cannot sustain bending moments, the inclination of the cable at the support gives the inclination of the support reaction, hence:

$$H_A = \frac{4}{\tan 14} = 16\ kN$$

So taking a cut in the cable at midspan, and taking moments about the cut, the sag is found as:

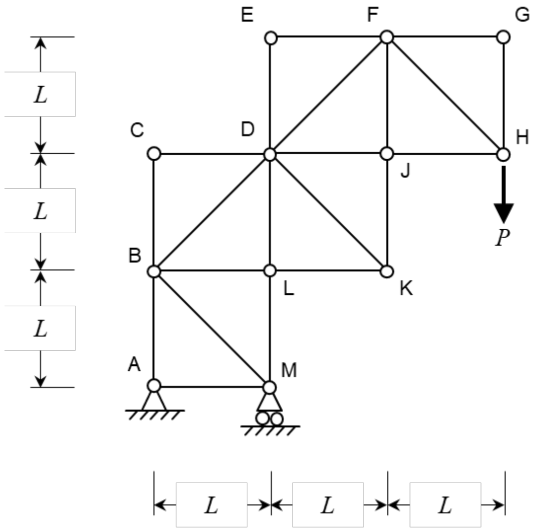


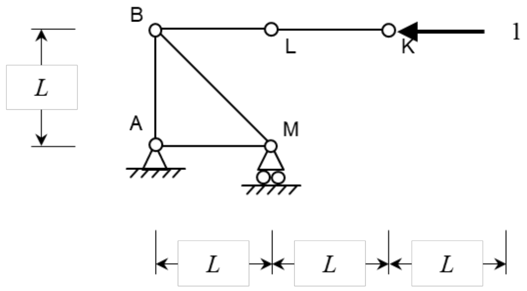
$$x = \frac{-\frac{1-0}{2} \times 6 \times \frac{6}{3} + 4 \times 6}{16} = 1.125\ m$$

Examiner's comments…

"Deflection of a hanging cable subject to a triangularly distributed load. Done well by many, but a surprising number of candidates, having correctly calculated the supported reactions and made a plausible attempt at a free-body diagram, were unable to then calculate the correct sag. A few candidates did not recognise that a cable cannot sustain bending moments and were thus on the wrong track from the start."

Q5 Long





a) Statically determinate truss structure. By equilibrium and free-bodies (method of joints and/or sections), or graphically, we obtain column 2:

	T	L	e [/EA]	T*	eT* [/EA]
AB	2P	L	PL	-1	-2PL
AM	0	L	0	-1	0
BC	0				
BD	2√2P				
BL	-2P	L	-2PL	-1	2PL
BM	0	√2L	0	√2	0
CD	0				
DE	0				
DF	√2P				
DJ	-P				
DK	2√2P				
DL	-3P				
EF	0				
FG	0				
FH	√2P				
FJ	-2P				
GH	0				
HJ	-P				
JK	-2P				
KL	-2P	L	-2PL	-1	2PL
LM	-3P				
				ΣeT*	2PL/EA

b) By virtual work. Considering the real forces (column 2) and member lengths (column 3), we obtain real extensions in column 4. Applying a virtual horizontal load at K (the joint of interest)) we obtain column 5. Summing the product of the real extensions and the virtual tensions we obtain the real deflection at the bottom of column 6. Alternatively, the solution can be obtained graphically by displacement diagram.

Note that there is no need to spend time calculating extensions for members that have zero force due to the virtual load (which in this case is every bar above the level of joint K).

c) Members DL and LM are most onerously loaded, and in compression. So check yielding: and buckling:

$$3P = \sigma_y A = 275 \times \pi \left[\left(\frac{101.6^2}{2} \right) - \left(\frac{93.6^2}{2} \right) \right] = 337\ kN$$

$$\therefore P_y = \frac{337}{3} = 112\ kN$$

and buckling:

$$3P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 200 \times 10^3 \times \frac{\pi}{4} \left[\left(\frac{101.6^4}{2} - \frac{93.6^4}{2} \right) \right]}{4000^2} = 180\ kN$$

$$\therefore P_e = 60\ kN$$

$P_e < P_y$, so buckling of members DL and LM at 60 kN govern.

Examiner's comments...

"A truss cantilever of sorts, subject to a point load.

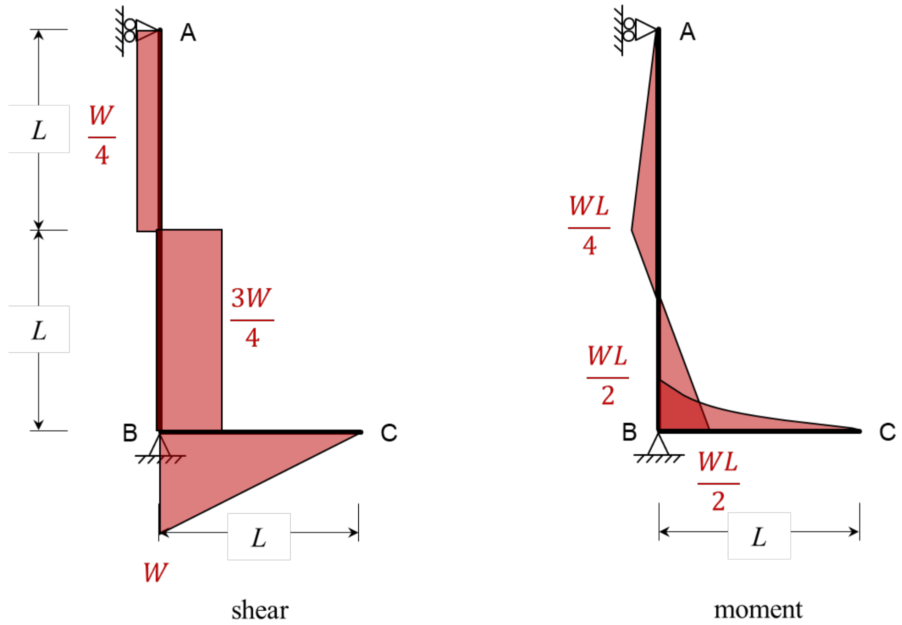
a) Done well by most candidates, with most or all forces correctly identified.

b) Done well by many candidates – most often by virtual work, but with a number of graphical solutions also in evidence. A number of candidates did not recognise that the virtual horizontal force at the node in question would only lead to forces in the lower portion of the structure.

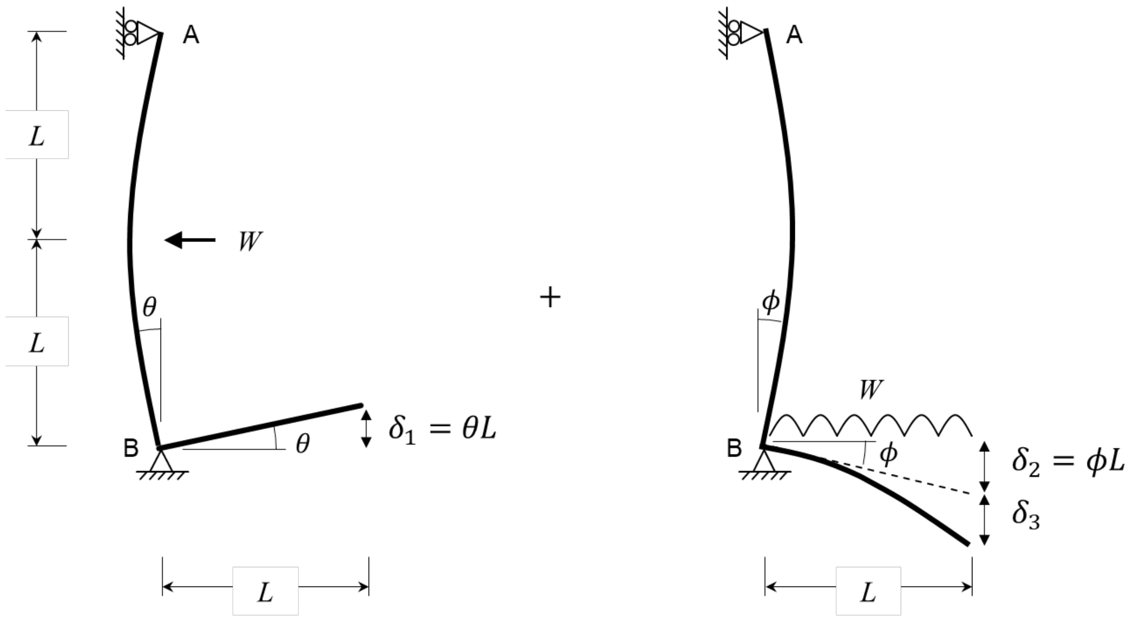
c) Well done by some. The need to check buckling was missed by many. A surprising number of candidates endeavoured to calculate the bending resistance of the cross section, despite the forces in a pin jointed truss being axial. Some did not consider the fact that the value of P applied to the structure leads to forces greater than P in many of the members."

Q6 Long

a) Statically determinate so solvable by equilibrium. Determine support reactions and then take free bodies as needed.

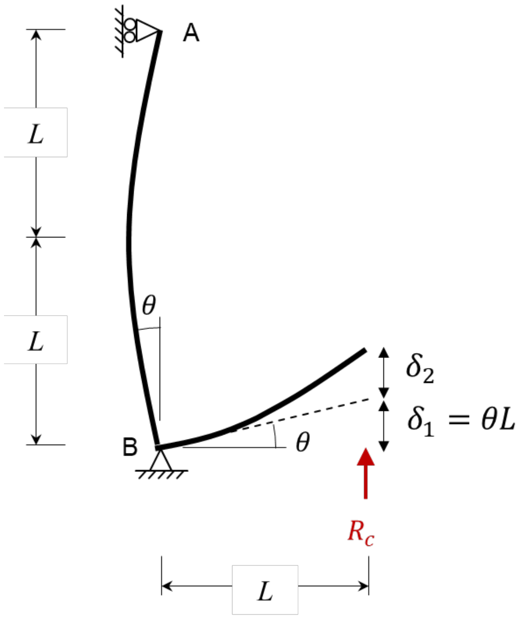


b) By superposition of databook cases, considering the effects of: the anti-clockwise rotation at B due to the point load; the clockwise rotation of B due to the distributed load, and the bending of BC due to the distributed load:



$$\delta_C = -\delta_1 + \delta_2 + \delta_3 = -\left[\frac{W(2L)^2}{16EI} \times L\right] + \left[\frac{\left(\frac{WL}{2}\right)2L}{3EI} \times L\right] + \left[\frac{WL^3}{8EI}\right] = \frac{5WL^3}{24EI} \ \textit{downward}$$

c) The introduction of a roller support at C renders the structure one degree indeterminate. The problem cannot be solved purely by equilibrium. However, we have one additional useful piece of information – that the roller at C prevents the vertical deflection of C. This means that deflection at C must be zero and, hence, that the reaction R_C required for compatibility must be exactly that needed to deflect the original determinate structure upward at C by the same magnitude that the original external loads caused it to deflect downward.



$$\delta_{Rc} = -\delta_1 + \delta_2 = \left[\frac{(R_C L)2L}{3EI} \times L\right] + \left[\frac{R_C L^3}{3EI}\right] = \frac{R_C L^3}{EI} = \delta_C = \frac{5WL^3}{24EI}$$

$$\therefore R_C = \frac{5W}{24EI}\ upwards$$

Examiner's comments…

"An L-shaped elastic beam subject to a point load and a distributed load.

a) Although some did this well, many did not. Many candidates do not appear to be comfortable finding reactions and then drawing shear and moment diagrams for a relatively simple determinate structure.

b) Most candidates recognised the need to superimpose databook cases to determine the deflection – but most did so unsuccessfully. Typically, only two cases were considered instead of the required three.

c) Most candidates recognised the need to use compatibility to find the reaction force in the newly indeterminate structure – although most again did not use the correct superposition of cases. Some candidates attempted to solve purely by equilibrium – which is sadly not possible."

1P2 Materials 2024 Crib

Question7

(a)

Let E_f and E_m be the Young's modulus of glass and polypropylene. Further let V_f be the volume fraction of the glass. (i) For Long parrallel fibres, the Young's modulus of the composites E_c is given by:

$$E_\mathrm{c} \;=\; V_\mathrm{f} E_\mathrm{f} \;+\; \bigl(1 - V_\mathrm{f}\bigr) E_\mathrm{m}$$

where $V_\mathrm{f} = 0.10$, $E_\mathrm{f} = 70$ GPa, and $E_\mathrm{m} = 1$ GPa. Substituting,

$$E_\mathrm{c} = 0.10 \times 70 \;+\; 0.90 \times 1 = 7.9\,\mathrm{GPa}.$$

(ii) Small particles:

$$\frac{1}{E_\mathrm{c}} \;=\; V_\mathrm{f}\,\frac{1}{E_\mathrm{f}} \;+\; \bigl(1 - V_\mathrm{f}\bigr)\,\frac{1}{E_\mathrm{m}}.$$

Substituting the same parameters,

$$\frac{1}{E_\mathrm{c}} = 0.10 \times \frac{1}{70} \;+\; 0.90 \times \frac{1}{1} = 0.0014286 + 0.90 = 0.9014286\,\mathrm{GPa}$$

which gives

$$E_\mathrm{c} \approx 1.11\,\mathrm{GPa}.$$

b

The axial strain is of the element is

$$\epsilon_1 = \frac{\sigma}{E},$$

and the lateral strains are

$$\epsilon_2 = \epsilon_3 = -\,\nu\,\frac{\sigma}{E}.$$

For small strains, the volumetric strain (dilatation) Δ is the sum of these normal strains:

$$\Delta = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{\sigma}{E} - 2\,\nu\,\frac{\sigma}{E} = \frac{\sigma}{E}\,\bigl(1 - 2\nu\bigr).$$

The volume is conserved if $\Delta = 0$, which implies

$$1 - 2\nu = 0 \quad\Longrightarrow\quad \nu = \frac{1}{2}.$$

Question 8

(a)

The main mechanisms are

1. Grain Boundary Strengthening
2. Solid Solution Strengthening
3. Work Hardening
4. Precipitation Strengthening

(b)

(i)
If each dislocation extends through the entire cube length L, then the total dislocation length in the sample is

$$\text{Total Length} = \rho \times L^3.$$

Annealed Sample

$$\text{Total Length}_{\text{annealed}} = 10^5 \times 10^3 = 10^8\,\text{mm}.$$

Cold-Worked (Hard) Sample

$$\text{Total Length}_{\text{cold-worked}} = 10^9 \times 10^3 = 10^{12}\,\text{mm}.$$

(ii)
For a square array of dislocations, the spacing d is approximately

$$d = \frac{1}{\sqrt{\rho}}.$$

Hence, for $\rho_{\text{cold-worked}} = 10^9\,\text{mm}^{-2}$:

$$d = \frac{1}{\sqrt{10^9}} = 10^{-4.5}\,\text{mm} = 3.16 \times 10^{-5}\,\text{mm} \approx 31.6\,\text{nm}.$$

(iii)
The flow stress contribution from dislocations is

$$\sigma_\rho \propto G\,b\,\sqrt{\rho},$$

where G is the shear modulus and b is the Burgers vector. Since σ_ρ scales with $\sqrt{\rho}$, the ratio of the cold-worked sample to the annealed sample is

$$\frac{\sigma_{\rho_{\text{cold-worked}}}}{\sigma_{\rho_{\text{annealed}}}} = \sqrt{\frac{\rho_{\text{cold-worked}}}{\rho_{\text{annealed}}}} = \sqrt{\frac{10^9}{10^5}} = \sqrt{10^4} = 100.$$

Thus, the dislocation contribution to strength in the cold-worked sample is approximately 100 times that of the annealed sample.

1P2 Materials - Solutions

Q9 (short)

(a) Mass of one C_2H_4 molecule:
$m = (2 \times 12.01 + 4 \times 1.008) \times 10^{-3}/(6.022 \times 10^{23}) = 4.658 \times 10^{-26}$ kg
where $N_A = 6.022 \times 10^{23}$ atoms / mol

There are two complete C_2H_4 molecules per unit cell, as sketched. The mass of each unit cell is therefore $2m$.

The volume of the unit cell is:
$V = (0.74 \times 10^{-9})(0.49 \times 10^{-9})(0.25 \times 10^{-9}) = 9.065 \times 10^{-29}$ m^3

The density is: $\rho = 2m/V = 1028$ kg m^{-3} [4]

This value will be an upper bound for practical polyethylene, as it assumes an idealised fully crystalline structure, with all molecular chains fully aligned. In practice the material will be semi-crystalline, with amorphous regions reducing the average density.

(b) (i) Semi-crystalline thermoplastic: Would be most likely used **above** its glass transition temperature. A degree of crystallinity would mean the polymer is glassy and brittle below T_g, but would retain a reasonable modulus above T_g. [2]

(ii) Amorphous thermoplastic: Would be most likely used **below** its glass transition temperature. The modulus drops rapidly above T_g for amorphous polymers, tending towards viscous flow. [2]

(iii) Natural rubber: Would be most likely used **above** its glass transition temperature. Rubber relies on cross linking rather than van der Waals bonds to provide the rubbery elastic behaviour. Below T_g the material would be glassy and stiff. [2]

Q10 (short)

(a) Let

- the container masses (kg) be m_{Al} and m_{PET}
- the recycled fractions be f_{Al} and f_{PET}
- the embodied energies (MJ/kg) be E_{Al} and E_{PET}
- the container volumes (m³) be V_{Al} and V_{PET}

(i) Embodied energy per unit volume contained if containers use only virgin material.
PET bottle: $W_{PET} = m_{PET}E_{PET,V}/V_{PET} = (30 \times 10^{-3})(84 \times 10^{3})/500 = 5.04$ kJ / ml
Aluminium can: $W_{Al} = m_{Al}E_{Al,V}/V_{Al} = (12 \times 10^{-3})(200 \times 10^{3})/330 = 8.00$ kJ / ml [2]

(ii) PET bottle contains 30% recycled polymer. Embodied energy per unit volume contained:

$$W_{PET} = \frac{m_{PET}}{V_{PET}}\left[(1 - f_{PET})E_{PET,V} + f_{PET}E_{PET,R}\right]$$
$$= (30 \times 10^{-3})(0.7 \times 84 \times 10^{3} + 0.3 \times 39 \times 10^{3})/500 = 4.23 \text{ kJ / ml}$$

Break even fraction of recycled aluminium:

$$W_{PET} = \frac{m_{Al}}{V_{Al}}\left[(1 - f_{Al})E_{Al,V} + f_{Al}E_{Al,R}\right]$$
$$\therefore\ f_{Al} = \frac{W_{PET}\left(\frac{V_{Al}}{m_{Al}}\right) - E_{Al,V}}{E_{Al,R} - E_{Al,V}} = \frac{4.23\left(\frac{300}{12 \times 10^{-3}}\right) - (200 \times 10^{3})}{(25 \times 10^{3}) - (200 \times 10^{3})} = 0.539 \qquad [4]$$

(b) Possible strategies to reduce the embodied energy of the aluminium alloy cans, per unit volume of water contained (any two):

- Increase the volume of the cans, e.g. to 500 ml, to reduce the amount of aluminium needed per unit volume of water contained. For efficient manufacturability, this would have to be a standard size and shape of can, which will have established processing routes and supply chains.
- Increase the volume fraction of recycled Al. This has a major impact due to the large differences between the embodied energies of virgin and recycled aluminium. Challenges include the impacts of contamination on the alloy properties and processability, and availability of suitable material through the material recycling supply chains.
- Reduce the thickness of the Al cans. Technical challenges include ensuring sufficient strength of the alloy to avoid failure in service (e.g. due to internal pressure of a sparkling drink, transport and handling loads), and processing challenges of forming very thin walled cans.

[2] + [2]

Question 11

(a) (i) Objective, maximise: $\dot{Q} = (T - T_0)2\pi\lambda\frac{D}{t}$

Constraint: $\sigma_h = \frac{pD}{2t} \leq \sigma_y$

Eliminate the free variable t from the objective using the constraint:

$$\dot{Q} = \frac{1}{p}(T - T_0)4\pi\lambda\sigma_y$$

Performance index to maximise: $M = \lambda\sigma_y$ [3]

Evaluate the performance index for each material:

- Cu alloy: $\lambda\sigma_y = 64.4 \times 10^9$ Pa W m^{-1} K^{-1}
- St steel: $\lambda\sigma_y = 11.0 \times 10^9$ Pa W m^{-1} K^{-1}
- Al alloy: $\lambda\sigma_y = 48.0 \times 10^9$ Pa W m^{-1} K^{-1}

So, **Cu alloy** is the best choice. [2]

Required wall thickness, given by constraint: $t \geq \frac{pD}{2\sigma_y}$ =0.163 mm [1]

(ii) Self weight: $\omega = \pi D t \rho g$ N/m

Second moment of area: $I = \pi D^3 t/8$

Mid-span deflection (structures data book):

$$\delta = \frac{5\omega L^4}{384EI} = \frac{40}{384}\frac{gL^4}{D^2}\frac{\rho}{E}$$ [3]

Evaluate this for each material

- Cu alloy: $\delta = 1.65$ mm **fails**
- St steel: $\delta = 1.03$ mm **passes**
- Al alloy: $\delta = 0.942$ mm **passes**

Constraint 2 eliminates Cu alloy, so **Al alloy** is now the best choice (best performance index). [2]

Required wall thickness is now: $t \geq \frac{pD}{2\sigma_y}$ =0.125 mm

Reduction in heat transfer: $\frac{\dot{Q}_{Al}}{\dot{Q}_{Cu}} = \frac{\lambda_{Al}}{\lambda_{Cu}}\frac{t_{Cu}}{t_{Al}} = 0.75$ [3]

(iii) Other constraints to consider (any two):

- Fracture toughness: A lower toughness may require a larger pipe wall thickness, to reduce risk of fast fracture, or fatigue crack growth. This would reduce the pipe performance.
- Oxidation and corrosion resistance: May restrict alloy choice, dependent on the nature of the high temperature gasses in the pipe, which could affect thermal performance.
- Manufacturability: Very small wall thickness are required, so material needs enough ductility to be able to be shaped into a very thin walled tube. Could restrict alloy choice, or require an increase in tube wall thickness.

[2] + [2]

(b)

(i) For the case $P > 0,\ T = T_0$:

Stresses in the pipe are due to the pressure alone:

$$\sigma_{hp} = \frac{pR}{t} \qquad \sigma_{lp} = 0 \quad \text{(given)}$$

No thermal strains. The total strain is therefore the elastic strain in the pipe wall:

$$\varepsilon_{hp}^{e} = \frac{pR}{E_p t} \qquad \varepsilon_{lp}^{e} = -\nu_p \frac{pR}{E_p t}$$

Because the coating is thin, the total (elastic) strain in the coating has to match:

$$\therefore \varepsilon_{hc}^{e} = \varepsilon_{hp}^{e} = \frac{pR}{E_p t} \qquad \varepsilon_{lc}^{e} = \varepsilon_{lp}^{e} = -\nu_p \frac{pR}{E_p t}$$ [4]

(ii) For the case $P > 0,\ T > T_0$:

Total strains are now a superposition of elastic strains and thermal strains. The elastic strains in the pipe are as above (the stresses in the pipe wall are the same). The total strains are therefore:

$$\varepsilon_{hp}^{tot} = \frac{pR}{E_p t} + \alpha_p (T - T_0) \qquad \varepsilon_{lp}^{tot} = -\nu_p \frac{pR}{E_p t} + \alpha_p (T - T_0)$$

Because the coating is thin, the total strain in the coating has to match this:

$$\varepsilon_{hc}^{tot} = \varepsilon_{hp}^{tot} = \frac{pR}{E_p t} + \alpha_p (T - T_0) \qquad \varepsilon_{lc}^{tot} = \varepsilon_{lp}^{tot} = -\nu_p \frac{pR}{E_p t} + \alpha_p (T - T_0)$$

The elastic strain in the coating is the total strain minus the thermal strain in the coating:

$$\varepsilon_{hc}^{e} = \varepsilon_{hc}^{tot} - \varepsilon_{hc}^{th} = \frac{pR}{E_p t} + \alpha_p (T - T_0) - \alpha_c (T - T_0)$$

$$\varepsilon_{lc}^{e} = \varepsilon_{lc}^{tot} - \varepsilon_{lc}^{th} = -\nu_p \frac{pR}{E_p t} + \alpha_p (T - T_0) - \alpha_c (T - T_0)$$

The hoop stress in the coating can be calculated from the elastic strains (Structures data book):

$$\sigma_{hc} = \frac{E_c}{1 - \nu_c^2} (\varepsilon_{hc}^{e} + \nu_c \varepsilon_{lc}^{e})$$

Substitute for the elastic strains:

$$\sigma_{hc} = \frac{E_c}{1 - \nu_c^2} \left[\frac{pR}{E_p t} (1 - \nu_c \nu_p) + \alpha_p (T - T_0)(1 + \nu_c) - \alpha_c (T - T_0)(1 + \nu_c) \right]$$

[8]

Question 12

(a) For most engineering alloys, a plot of the logarithm of crack growth per cycle da / dN versus the logarithm of the stress intensity factor range ΔK exhibits a sigmoidal shape as shown in the figure below. Three distinct regions, labelled I, II, and III are identified.

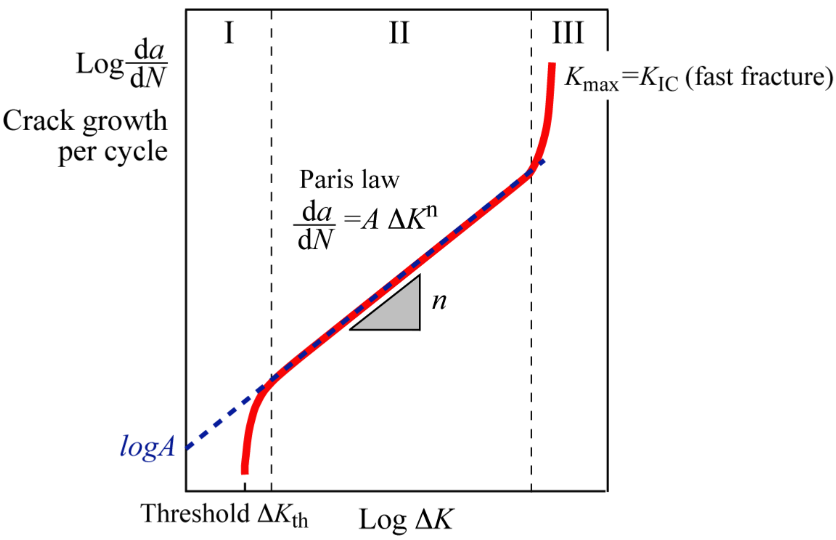
Region I – Crack Initiation: Crack growth per cycle is zero below a threshold cyclic stress intensity factor range ΔK_{th}.

Region II - Steady- State Crack Propagation described by the Paris law:

$$\frac{da}{dN} = A \ \Delta K^n$$

where A and n are constants – see figure below.

Region III – Fast Fracture: At high ΔK , crack growth rate increases rapidly. As K_{max} approaches K_{IC}, fast fracture occurs.



(b)

(i)
The hoop stress σ is

$$\sigma = \frac{p\,R}{t} == \frac{(4.1\,\mathrm{MPa}) \times (4\,\mathrm{m})}{0.04\,\mathrm{m}} = 410\,\mathrm{MPa}.$$

The critical crack size for fast fracture can be computed as

$$K_{\mathrm{IC}} = \sigma\sqrt{\pi\,c_{\mathrm{crit}}} \implies c_{\mathrm{crit}} = \frac{1}{\pi}\left(\frac{K_{\mathrm{IC}}}{\sigma}\right)^2.$$

Substitute $\sigma = 410\,\mathrm{MPa}$, $K_{\mathrm{IC}} = 200\,\mathrm{MPa}\sqrt{\mathrm{m}}$:

$$\frac{K_{\mathrm{IC}}}{\sigma} = \frac{200}{410} \approx 0.4878, \quad (0.4878)^2 \approx 0.238, \quad c_{\mathrm{crit}} \approx \frac{0.238}{\pi} \approx 0.0758\,\mathrm{m} = 75.8\,\mathrm{mm}.$$

Since $c_{\mathrm{crit}} > t$, the crack will penetrate through the wall (causing a leak) *before* it reaches the critical size needed for fast fracture. Therefore te vessel will fail by leaking.

(ii)

The vessel experiences 2000 pressurization cycles, during which the crack grows from an *initial* size c_i to the *through-wall* size $c_f = t = 0.04\,\mathrm{m}$. We use the Paris law:

$$\frac{dc}{dN} = A\,(\Delta K)^n, \quad \Delta K = \Delta\sigma\sqrt{\pi\,c}.$$

Given:

$$n = 4, \quad A = 2.44 \times 10^{-14}\,\mathrm{MPa}^{-4}\mathrm{m}^{-1}, \quad \Delta\sigma \approx \sigma_{\max} = 410\,\mathrm{MPa}.$$

Step A: Express Paris law in terms of c:

$$\Delta K = \Delta\sigma\sqrt{\pi\,c} \implies (\Delta K)^4 = (\Delta\sigma)^4(\pi\,c)^2.$$

Thus:

$$\frac{dc}{dN} = A\,(\Delta\sigma)^4\,(\pi\,c)^2 = A\,(410)^4\,\pi^2\,c^2.$$

Define

$$B = A\,(410)^4\,\pi^2 \implies \frac{dc}{dN} = B\,c^2.$$

Step B: Integrate from c_i to c_f over 2000 cycles:

$$\int_{c_i}^{c_f} \frac{dc}{c^2} = \int_0^{2000} B\,dN.$$

Left side:

$$\int_{c_i}^{c_f} \frac{dc}{c^2} = \left[-\tfrac{1}{c}\right]_{c_i}^{c_f} = \frac{1}{c_i} - \frac{1}{c_f}.$$

Right side:

$$\int_0^{2000} B\,dN = 2000\,B.$$

Hence

$$\frac{1}{c_i} - \frac{1}{c_f} = 2000\,B \implies \frac{1}{c_i} = \frac{1}{c_f} + 2000\,B.$$

Step C: Compute B and solve for c_i.

$$(410)^4 \approx 2.83 \times 10^{10}, \quad \pi^2 \approx 9.8696, \quad A = 2.44 \times 10^{-14}.$$

So

$$B \approx 2.44 \times 10^{-14} \times 2.83 \times 10^{10} \times 9.8696 \approx 0.0068\,\mathrm{m}^{-1}\,\mathrm{cycle}^{-1}.$$

Then

$$2000\,B \approx 13.6.$$

Using $c_f = 0.04\,\mathrm{m}$:

$$\frac{1}{c_i} = \frac{1}{0.04} + 13.6 = 25 + 13.6 = 38.6 \implies c_i = \frac{1}{38.6} \approx 0.0259\,\mathrm{m} = 25.9\,\mathrm{mm}.$$

$$\boxed{c_i \approx 26\,\mathrm{mm}.}$$

Any initial crack of this size will reach the full 40 mm thickness in 2000 cycles.

(iii)

We perform a single high-pressure test so that a crack of length c_i reaches K_{IC}. Any crack longer than c_i fails in the test.

$$K_{\mathrm{IC}} = \sigma_{\mathrm{proof}}\sqrt{\pi\,c_i}, \quad \sigma_{\mathrm{proof}} = \frac{p_{\mathrm{proof}}\,R}{t}.$$

Hence:

$$p_{\mathrm{proof}} = \frac{K_{\mathrm{IC}}\,t}{R\,\sqrt{\pi\,c_i}}.$$

Substitute:

$$K_{\mathrm{IC}} = 200\,\mathrm{MPa}\sqrt{\mathrm{m}}, \quad t = 0.04\,\mathrm{m}, \quad R = 4\,\mathrm{m}, \quad c_i \approx 0.026\,\mathrm{m}.$$

Inside the root: $\pi \times 0.026 \approx 0.0817$, $\sqrt{0.0817} \approx 0.286$. Then

$$p_{\text{proof}} \approx \frac{(200 \times 0.04)\,\text{MPa}\sqrt{\text{m}}}{4\ \times 0.286} \approx \frac{8\,\text{MPa}\sqrt{\text{m}}}{1.144} \approx 7.0\,\text{MPa}.$$

$$\boxed{p_{\text{proof}} \approx 7.0\,\text{MPa}.}$$