ENGINEERING TRIPOS PART IA 2013

Paper 4 Mathematical Methods

Solutions

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2013 IA Paper 4 Crib

1. $$e^{z} = \frac{15e^{i\pi/12} - 10}{3+4i}$$

$$= \frac{15\cos \pi/12 + i\,15\sin \pi/12 - 10}{3+4i}$$

$$= 1.160 - .252i$$

$$= 1.187\, e^{-.214i}$$

i.e. $$e^{x+iy} = 1.187\, e^{-.214i}$$

$$\Rightarrow \quad e^{x} = 1.187 \quad \Rightarrow \quad x = .171$$

and $$y = -.214 + 2n\pi \quad n \text{ integer}$$

$$\therefore \quad \underline{z = .171 - .214i + 2n\pi i}$$

2 (a) $f(x) = \sin^2 x - x^2$ $\qquad g(x) = 1 - \cos x$

$f(0) = 0 \qquad g(0) = 0$

$f'(x) = 2\sin x\cos x - 2x \qquad f'(0) = 0 \qquad g'(x) = \sin x \qquad g'(0) = 0$

$f''(x) = 2\cos^2 x - 2\sin^2 x - 2 \qquad f''(0) = 0 \qquad g''(x) = \cos x \qquad g''(0) = 1$

$$\therefore \lim_{x \to 0} \frac{f(x)}{g(x)} = 0$$

(b)
$$\frac{1}{1-\cos x} - \frac{2}{x^2} = \frac{1}{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)} - \frac{2}{x^2}$$

$$= \frac{1}{\frac{x^2}{2}\left[1 - \frac{x^2}{12} + \cdots\right]} - \frac{2}{x^2}$$

$$= \frac{2}{x^2}\left[1 + \frac{x^2}{12} + \cdots\right] - \frac{2}{x^2} \quad \text{using Binomial}$$

$$= \frac{2}{12} + \cdots$$

$$\therefore \lim_{x \to 0}\left(\frac{1}{1-\cos x} - \frac{2}{x^2}\right) = \frac{1}{6}$$

3.

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} \frac{5}{4}-\lambda & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{5}{4}-\lambda\right)^2 - \left(\frac{3}{4}\right)^2 = 0 \quad \Rightarrow \quad \lambda - 5/4 = \pm 3/4$$

ie. $\lambda = \frac{8}{4}$ or $\frac{2}{4}$ ie. $\underline{\lambda = 2 \text{ or } \frac{1}{2}}$

(i) $\lambda = 2$ $\Rightarrow \left(\frac{5}{4}-2\right)x + \frac{3}{4}y = 0 \Rightarrow -\frac{3x}{4} + \frac{3y}{4} = 0$

e-vector $= \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow$ $\underline{\text{e-vector } \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ (or any multiple of it)

(ii) $\lambda = \frac{1}{2}$ $\Rightarrow \left(\frac{5}{4}-\frac{1}{2}\right)x + \frac{3}{4}y = 0 \Rightarrow \frac{3x}{4} + \frac{3y}{4} = 0$

$\Rightarrow$ e-vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (or any multiple of it)

4

(a) Let $\underline{c} = [x, y, z]^T$

$$\Rightarrow \quad x\begin{bmatrix}1\\0\\0\end{bmatrix} + \left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right)\begin{bmatrix}\frac{1}{2}\\ \sqrt{3}/2\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2\sqrt{3}}\\ 0\end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} x + \frac{x}{4} + \frac{\sqrt{3}y}{4} &= \frac{1}{2} \\ \frac{\sqrt{3}x}{4} + \frac{3y}{4} &= \frac{1}{2\sqrt{3}} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} 5x + \sqrt{3}y &= 2 \\ x + \sqrt{3}y &= 2/3 \end{aligned}$$

$$\therefore 4x = \frac{4}{3} \Rightarrow \underline{x = \frac{1}{3}} \quad \text{and} \quad \sqrt{3}y = \frac{1}{3} \Rightarrow \underline{y = \frac{1}{3\sqrt{3}}}$$

Then $|\underline{c}| = 1 \Rightarrow \frac{1}{9} + \frac{1}{27} + z^2 = 1 \Rightarrow z^2 = \frac{23}{27}$

i.e. $\underline{\underline{c} = \left[\frac{1}{3}, \frac{1}{3\sqrt{3}}, \pm\sqrt{\frac{23}{27}}\right]^t}$

(b) $\underline{p} = \lambda \underline{a} = \begin{bmatrix}\lambda\\0\\0\end{bmatrix}$. Let $\underline{c} = [x, y, z]^T$

$$\therefore \begin{bmatrix}\lambda\\0\\0\end{bmatrix} = x\begin{bmatrix}1\\0\\0\end{bmatrix} + \left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right)\begin{bmatrix}\frac{1}{2}\\ \sqrt{3}/2\\ 0\end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} \lambda &= x + \frac{x}{4} + \frac{\sqrt{3}y}{4} \\ 0 &= \frac{x}{2} + \frac{\sqrt{3}y}{2} \end{aligned}\right\} \Rightarrow \quad y = -\frac{x}{\sqrt{3}}$$

$\therefore \lambda = x = |\underline{p}|$ (or rather $|\underline{p}| = |x|$)

$\underline{c}$ unit $\Rightarrow x^2 + y^2 + z^2 = 1 \Rightarrow x^2 + \frac{x^2}{3} + z^2 = 1$

Maximise x such that $\frac{4x^2}{3} + z^2 = 1 \Rightarrow z = 0 \quad x^2 = 3/4$

$$\therefore \underline{\text{Max } |\underline{p}| = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}}$$

5 $\quad \ddot{y} + 4\dot{y} + 9y = 1 + \sin\omega t$

(a) C.F. $y \propto e^{\lambda t} \Rightarrow \lambda^2 + 4\lambda + 9 = 0 \Rightarrow \lambda = \dfrac{-4 \pm \sqrt{16-36}}{2}$

$$= -2 \pm i\sqrt{5}$$

$$\therefore \; y_{CF} = \left(A\cos\sqrt{5}t + B\sin\sqrt{5}t\right)e^{-2t}$$

P.I. Try $y = C + D\sin\omega t + E\cos\omega t$

$$\dot{y} = +D\omega\cos\omega t - E\omega\sin\omega t$$

$$\ddot{y} = -D\omega^2\sin\omega t - E\omega^2\cos\omega t$$

$$\therefore \; -D\omega^2\sin\omega t - E\omega^2\cos\omega t + 4(D\omega\cos\omega t - E\omega\sin\omega t)$$

$$+ 9C + 9D\sin\omega t + 9E\cos\omega t = 1 + \sin\omega t$$

$\Rightarrow \quad C = \frac{1}{9}$ and

$$-D\omega^2 - 4E\omega + 9D = 1$$

$$-E\omega^2 + 4D\omega + 9E = 0$$

$$\therefore \; D = \frac{\omega^2 - 9}{4\omega}E$$

$$\& \quad -\frac{(\omega^2-9)^2}{4\omega}E - 4\omega E = 1 \quad \Rightarrow E = \frac{-4\omega}{16\omega^2 + (\omega^2-9)^2}$$

$$\& \quad D = \frac{-(\omega^2-9)}{(\omega^2-9)^2 + 16\omega^2}$$

$\therefore$ General solution

$$y = \frac{1}{9} - \frac{\omega^2-9}{\omega^4 - 2\omega^2 + 81}(\sin\omega t) - \frac{4\omega}{\omega^4 - 2\omega^2 + 81}\cos\omega t$$

$$+ e^{-2t}\left[A\cos\sqrt{5}t + B\sin\sqrt{5}t\right]$$

(b) When $\omega = 3$, no sine term and

$$y = \frac{1}{9} - \frac{1}{12}\cos 3t \quad + \quad \begin{matrix} C.F \\ e^{-2t}\left[A\cos\sqrt{5}t + B\sin\sqrt{5}t\right] \end{matrix}$$

y_{PI} is $90°$ out of phase with $\sin\omega t$ (i.e. $\sin 3t$)

(c) As $\omega \to \infty$

$$y \simeq \frac{1}{9} - \frac{\sin\omega t}{\omega^2} + y_{CF}$$

$$\left[\begin{matrix} \dfrac{\omega^2-9}{\omega^4-2\omega^2+81} \simeq \dfrac{\omega^2}{\omega^4} = \dfrac{1}{\omega^2} \\[2ex] \dfrac{4\omega}{\omega^4-2\omega^2+81} \simeq \dfrac{4\omega}{\omega^4} = \dfrac{4}{\omega^3} \end{matrix}\right]$$

y_{PI} is $180°$ out of phase with $\sin\omega t$

& amplitude $\to 0$ like $\frac{1}{\omega^2}$

6.

Take L.T. $0 = L(\ddot{y} + 2\dot{y} + 2y)$

$$= s^2 Y - s y_0 - \dot{y}_0 + 2(sY - y_0) + 2Y$$

$$= (s^2 + 2s + 2)Y - 1$$

$$\Rightarrow \quad Y = \frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}$$

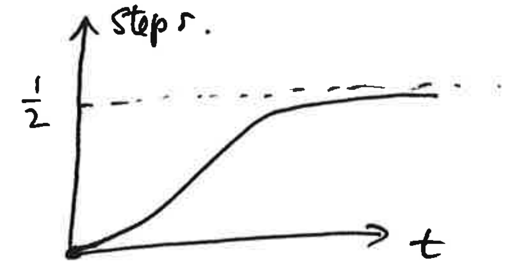
Inverting $\quad y = e^{-t} \sin t$

7.

(a) $y(0) = \dot{y}(0) = 0 \Rightarrow \frac{1}{2} + A + B = 0, \; -A - 2B = 0$

$\therefore \frac{1}{2} - 2B + B = 0 \Rightarrow \underline{B = \frac{1}{2}, \; A = -1}$

$\therefore$ step response $= \underline{\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}$

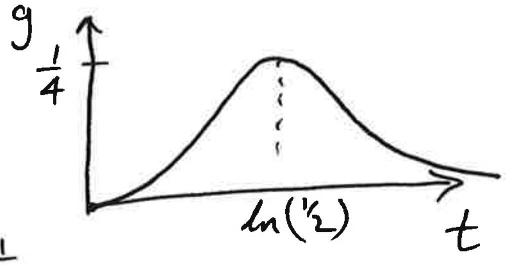


(b) $g(t) = \frac{d(\text{step r})}{dt} = \underline{e^{-t} - e^{-2t}}$

$g(t) = 0 \Rightarrow e^{-t} = e^{-2t}$

$\Rightarrow e^{-t} = 1 \Rightarrow t = 0$

$\therefore$ step response monotonic



$g'(t) = 0 \Rightarrow e^{-t} = 2e^{-2t} \Rightarrow e^{-t} = \frac{1}{2}$

$t = -\ln\left(\frac{1}{2}\right)$

$g_{max} = e^{+\ln(\frac{1}{2})} - e^{+2\ln(\frac{1}{2})}$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

(c) $y(t) = \int_{\tau=0}^{t} g(t-\tau)f(\tau)d\tau = \int_{\tau=0}^{t}\left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right]e^{-\tau}d\tau$

$= e^{-t}\int_{\tau=0}^{t} dt - e^{-2t}\int_{\tau=0}^{t} e^{\tau} d\tau$

$= te^{-t} - e^{-2t}\left[e^{t} - 1\right]$

$\begin{cases} = te^{-t} - e^{-t} + 1 & \text{for } t > 0 \\ = 0 & t < 0 \end{cases}$

(d)

$$y(t) = 0 \qquad t < 0$$

$$= t e^{-t} - e^{-t} - e^{-2t} \qquad \left. 0 \le t \le T \right\} \text{as in (c)}$$

For $t > T$

$$y(t) = e^{-t} \int_{\tau=0}^{T} d\tau - e^{-2t} \int_{\tau=0}^{T} e^{\tau} d\tau$$

$$= T e^{-t} - e^{-2t} (e^{T} - 1)$$

8.

Old machine

Expected selling price $= \sum \text{outcome} \times p(\text{outcome})$

$= 10 \times (.9) + 5 \times (.05) + 0 \times (.05)$

$= £\,9.25$

$\Rightarrow$ Expected profit $= 25p$

New machine

Make $n \Rightarrow £\,n$ profit

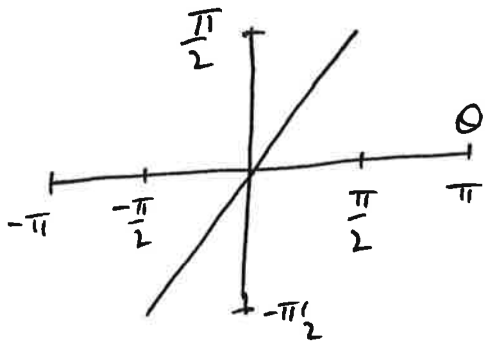
New machine paid from profit

$\Rightarrow$ Break even when

$\frac{n}{4} + 10{,}000 = n \quad \Rightarrow n = \frac{40{,}000}{3} = \underline{13{,}333}$

9.

$f(\theta) =$



Range $= 2\pi$

Odd fn $\Rightarrow$ sine's only $\Rightarrow$ $f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \theta \sin n\theta \, d\theta$

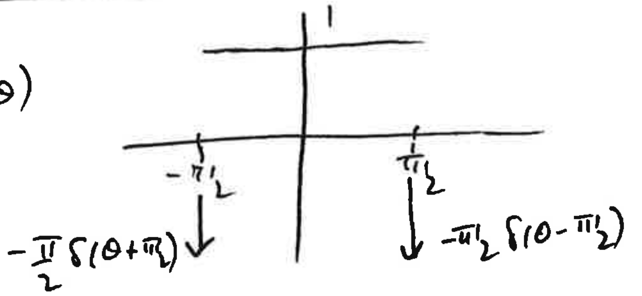
$$\frac{-2}{\pi}\left[-\frac{\theta\cos n\theta}{n}\right]_0^{\pi/2} + \frac{2}{\pi}\int_0^{\pi/2} 1 . \frac{\cos n\theta}{n} \, d\theta$$

$$= -\frac{\cos n\pi/2}{n} + \frac{2}{n\pi}\left[\frac{\sin n\theta}{n}\right]_0^{\pi/2}$$

$$= -\frac{\cos \frac{n\pi}{2}}{n} + \frac{2}{n^2\pi}\sin n\pi/2$$

Alter

$f'(\theta)$



Even fn

$f'(\theta) = d + \sum_1^{\infty} a_n \cos n\theta$

Area under curve $= -\pi + \pi = 0$

$\Rightarrow d = 0$

& $a_n = \frac{2}{\pi}\int_0^{\pi/2} 1 . \cos n\theta \, d\theta - \frac{2}{\pi} . \frac{\pi}{2}\int_0^{\pi} \delta(\theta)\cos n\theta \, d\theta$

$$= \frac{2}{\pi}\frac{\sin n\pi/2}{n} - \cos n\pi/2$$

$f'(\theta) = \sum a_n \cos n\theta \Rightarrow f(\theta) = \sum a_n \frac{\sin n\theta}{n} + d$ & $d = 0$

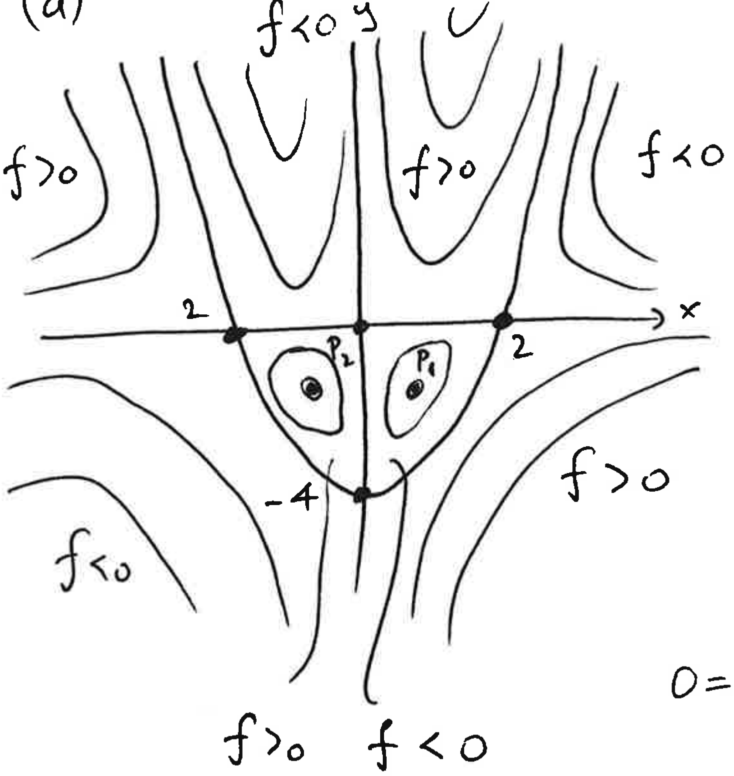
$\therefore f(\theta) = \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{1}{n}\cos\frac{n\pi}{2}\right]\sin n\theta$

10.

$$f(x,y) = xy(y+4-x^2) = xy^2 + 4xy - x^3y$$

$$f=0 \Rightarrow x=0,\ y=0 \text{ or } y = x^2-4$$

(a)



(b) Saddle points at (0,0), (2,0), (−2,0), (0,−4)

P_1 is min

P_2 is max

At stationary points

$$0 = \frac{\partial f}{\partial x} = y^2 + 4y - 3x^2y = y(y+4-3x^2)$$

$$0 = \frac{\partial f}{\partial y} = 2xy + 4x - x^3 = x(2y+4-x^2)$$

$\therefore$ $y=0$ OR $y+4-3x^2=0$

AND $x=0$ OR $2y+4-x^2=0$

(i) $y=0,\ x=0 \to (0,0)$ ✓

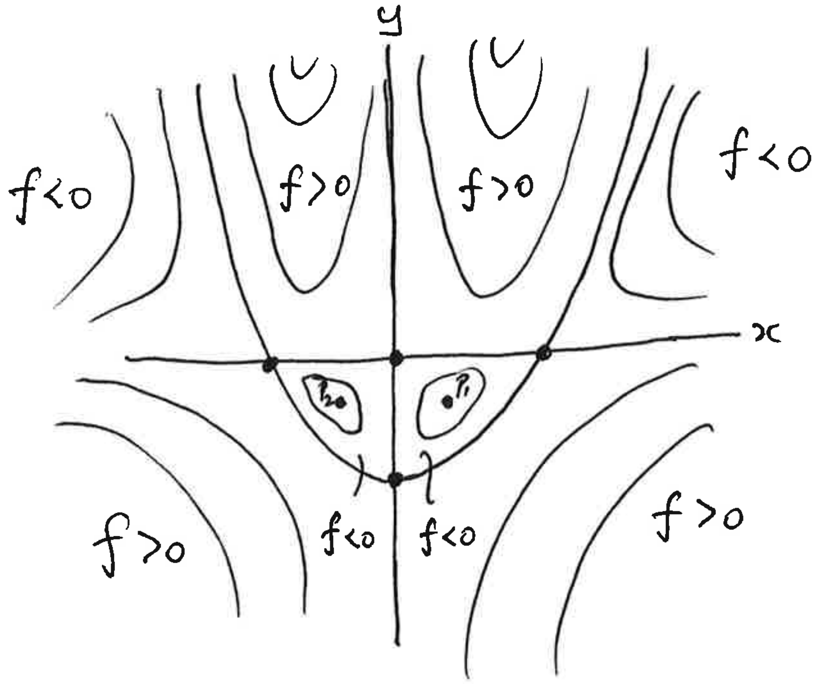
(ii) $y=0$ $2y = x^2-4$
$\Rightarrow x = \pm 2 \to (\pm 2, 0)$ ✓

(iii) $x=0$ $y+4-3x^2=0$
$\to (0,-4)$ ✓

(iv) $\left.\begin{array}{l} y = 3x^2-4 \\ 2y = x^2-4 \end{array}\right\}$ $5y = -8$ $y = -\frac{8}{5}$ & $3x^2 = 4+y$
$\Rightarrow 6y = 3x^2 - 12$
$= 4 - \frac{8}{5}$
$= \frac{12}{5}$
$x^2 = \frac{4}{5}$
$\therefore x = \pm\frac{2}{\sqrt{5}}$

$P_1 = \left(\frac{2}{\sqrt{5}}, -\frac{8}{5}\right)$ min P_2 is $\left(\frac{-2}{\sqrt{5}}, -\frac{8}{5}\right)$ max

(c) Zero contours unchanged, but sign changes about $x=0$ no longer happen & we now have symmetry about $x=0$



P_1 & P_2 are now minima

All others are saddles

11(a)

```
class building {
  public:
    int x_coord;
    int y_coord;
    string name;
    BuildingType btype;
    int uniq_id;
};

building indata[DATASIZE];
```

(b)

```
void map_insert(building); // add a building to the map

int build_map() {
    int i;
    for (i = 0; i < DATASIZE; i++) {
     indata[i].uniq_id = next_id();
     map_insert(indata[i]);
    }
   return (0);
}
```

12(a) `myfunc` searches to see if the value `val` is equal to any of the bottom four bytes in any of the integers in the array `data`, of length `ndata`, and returns the number of integers containing a matching byte. The outer loop scans through the integers in the array. The inner loop scans through the four bytes in the integer. The way in which `found` and `nfound` are used ensures that if an integer contains two matching bytes, only one is counted.

(b) The algorithm is O(N). (The inner loop does not depend on N.)