Answers to questions are shown at the end of this paper

ENGINEERING TRIPOS PART IA

Tuesday 11 June 2013   9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (**short**) Find all values of z which satisfy the equation:

$$10 + e^{z}\left(3+4i\right) = 15e^{i\pi/12}$$ [10]

2 (**short**)

(a) Using l'Hopital's rule, and not otherwise, find

$$\lim_{x \to 0} \left(\frac{\sin^2 x - x^2}{1 - \cos x} \right)$$ [5]

(b) Using a power series expansion, and not otherwise, find

$$\lim_{x \to 0} \left(\frac{1}{1 - \cos x} - \frac{2}{x^2} \right)$$ [5]

3 (**short**) The matrix $\mathbf{A}$ is

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of $\mathbf{A}$. [10]

(TURN OVER

4 (**long**) The vectors **a** and **b** are used to define a plane through the origin, where

$$\mathbf{a} = [1, 0, 0]^{\mathrm{T}} \quad \text{and} \quad \mathbf{b} = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right]^{\mathrm{T}}$$

The vector **c** is of unit magnitude, and satisfies the equation

$$(\mathbf{c}\cdot\mathbf{a})\mathbf{a} + (\mathbf{c}\cdot\mathbf{b})\mathbf{b} = \mathbf{p}$$

(a) For the case when the vector **p** is given by $\mathbf{p} = \left[\frac{1}{2}, \frac{1}{2\sqrt{3}}, 0\right]^{\mathrm{T}}$, find the corresponding vector **c** . Note that **c** is not necessarily in the plane. [18]

(b) If **p** is in the direction of **a**, what is the maximum magnitude of **p** as the direction of **c** varies (while remaining a unit vector) ? [12]

5 (**long**) The function $y(t)$ satisfies the equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 9y = 1 + \sin\omega t$$

where ω is a constant.

(a) Find the general solution for y. [14]

(b) Describe, carefully, the special features of the solution when $\omega = 3$ and evaluate the particular integral for this case. [12]

(c) *Without further calculation*, describe what happens to the particular integral as $\omega \to \infty$. [4]

(TURN OVER

SECTION B

6 (**short**) Using Laplace Transforms, solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0 \qquad \text{for } t > 0$$

where $y(0) = 0$ and $\dot{y}(0) = 1$. [10]

7 (**long**)

(a) The step response of a linear system is given by

$$\begin{cases} 0 & t < 0 \\ \frac{1}{2} + Ae^{-t} + Be^{-2t} & t \geq 0 \end{cases}$$

Find the values of A and B and sketch the step response. [5]

(b) Find and sketch the impulse response of the system. [5]

(c) Find the response of the system to

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$ [12]

(d) Find the response of the system to

$$f(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$ [8]

(TURN OVER

8 (**short**) A component costs £9 to make and can, if perfect, be sold for £10. The machine producing components is not reliable; 5% of components have to be scrapped and 5% can only be sold at half the price of a perfect one. A replacement machine, that produces a negligible number of faulty components, is purchased for £10,000. How many components need to be made on the new machine and then sold before there is an increase in profit over that which would have been obtained making the same number on the old machine ? [10]

9 (**short**) Find the Fourier Series representation of the function $f(\theta)$ over the range $-\pi \leq \theta \leq \pi$, where

$$f(\theta) = \begin{cases} 0 & -\pi \leq \theta < -\dfrac{\pi}{2} \\ \theta & -\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2} \\ 0 & \dfrac{\pi}{2} < \theta \leq \pi \end{cases}$$ [10]

10 (**long**)

(a) Sketch contours of the function

$$f(x,y) = x\,y\left(y + 4 - x^2\right)$$ [8]

(b) Find the stationary points of f and classify them using your sketch. [14]

(c) Sketch contours of the function

$$g(x,y) = x^2\,y\left(y + 4 - x^2\right)$$

and indicate on your sketch the approximate position and type of any stationary points. [8]

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SECTION C

11 (**short**) The code extracts shown in Figs. 1(a) and 1(b) are parts of a program which takes data about buildings to enter into a map. Unfortunately, this program was written by someone who did not know how to use C++ classes.

(a) Define a suitable class, `building`, to represent all the data in Fig. 1(a) and declare a variable, `indata`, as an array of elements of this class to hold the data. [5]

(b) Redefine the function prototype for `map_insert` from that given in Fig. 1(b) to use this new representation of the buildings' data. Rewrite the function `build_map` on a line by line basis to use this new representation. (The details of the implementation of the functions `map_insert` and `next_id` do not matter for this question. They operate as described in the comments in Fig. 1(b) and someone else will rewrite `map_insert` to conform to its new function prototype.) [5]

```
enum BuildingType { House, Hotel, Pub, Shop, School, Church };

const int DATASIZE = 1000;
int x_coord[DATASIZE];
int y_coord[DATASIZE];
string name[DATASIZE];
BuildingType btype[DATASIZE];
int uniq_id[DATASIZE];
```

Fig. 1(a)

```
int next_id();     // returns a unique id number for the building
void map_insert(int, int, string, BuildingType, int);
// add a building to the map

int build_map() {
  int i;
  for (i = 0; i < DATASIZE; i++) {
    uniq_id[i] = next_id();
    map_insert (x_coord[i], y_coord[i], name[i], btype[i],
                uniq_id[i]);
    }
  return (0);
}
```

Fig. 1(b)

12 (**short**)

(a) Explain the purpose of the function given in Fig. 2, briefly describing its operation. Note that the C++ % operator returns the remainder from a division and the / operator returns the quotient. [7]

(b) What is the algorithmic complexity of this function with respect to the number of elements in the input data array? Briefly explain your reasoning. [3]

```
int myfunc(unsigned int val, unsigned int data[], int ndata)
{
    int i, j;
    unsigned int d;
    bool found;
    int nfound = 0;

    for (i = 0; i < ndata; i++) {
      d = data[i];
      found = false;
      for (j = 0; found == true and j < 4; j++) {
        if ((d % 256) == val) {
          found = true;
        }
        d = d/256;
      }
      if (found == true) {
        nfound++;
      }
    }
    return (nfound);
}
```

Fig. 2

END OF PAPER

Answers to questions

1. $0.171 + i(-0.214 + 2\pi n)$

2. (a) 0 (b) 1/6

3. Eigenvalues 2 and ½. Eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ (or any multiples of them)

4. (a) $\left[\frac{1}{3}, \frac{1}{3\sqrt{3}}, \pm\sqrt{\frac{3}{27}}\right]^{\mathrm{T}}$ (b) $\frac{\sqrt{3}}{2}$

5. (a) $\frac{1}{9} - \frac{\omega^2 - 9}{\omega^4 - 2\omega^2 + 81}\sin\omega t - \frac{4\omega}{\omega^4 - 2\omega^2 + 81}\cos\omega t + e^{-t}\left(A\cos\sqrt{5}\,t + B\sin\sqrt{5}\,t\right)$

   (b) $\frac{1}{9} - \frac{1}{12}\cos 3t$

6. $e^{-t}\sin t$

7. (a) -1, ½ (b) $e^{-t} - e^{-2t}$ (c) $\begin{cases} 0 & t < 0 \\ (t-1)e^{-t} + e^{-2t} & 0 \le t \end{cases}$

   (d) $\begin{cases} 0 & t < 0 \\ (t-1)e^{-t} + e^{-2t} & 0 \le t \le T \\ Te^{-t} - (e^{T} - 1)e^{-2t} & t > T \end{cases}$

8. 13,334

9. $\sum_{n=1}^{\infty}\left(-\frac{1}{n}\cos\frac{n\pi}{2} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2}\right)\sin n\theta =$

$$\frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\sin(2m-1)\theta}{(2m-1)^2} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin 2n\theta}{n}$$

10. (b) Saddles at (0, 0), (±2, 0), (0, –4).

    Minimum at $\left(\frac{2}{\sqrt{5}}, -\frac{8}{5}\right)$, Maximum at $\left(-\frac{2}{\sqrt{5}}, -\frac{8}{5}\right)$

12. $O(ndata)$

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