Part IA, Paper 4: Mathematical Methods, 2022

F Cirak

9. (a) Cross-product of the two directional vectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Two parallel planes containing the two lines:

$$-x - 2y + 3z - 9 = 0\,, \quad -x - 2y + 3z - 6 = 0$$

$$d = \frac{3}{\sqrt{14}}$$

(b) i.

$$\boldsymbol{p}.(\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{c}.\boldsymbol{a})\,(\boldsymbol{a}.(\boldsymbol{a} \times \boldsymbol{b})) + (\boldsymbol{c}.\boldsymbol{b})\,(\boldsymbol{b}.(\boldsymbol{a} \times \boldsymbol{b})) = 0$$

ii. The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ span the entire space.

iii. Assume

$$\boldsymbol{c} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}$$

such that

$$\begin{bmatrix} c_x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} c_x + 2c_z \\ 0 \\ 2c_x + 4c_z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \quad \Rightarrow \quad c_x = -2\,, \quad c_z = 3 \quad \text{and} \quad c_y \in \mathbb{R}$$

Assessors' remarks: This question was, in general, very well answered. In part (a), most applicants could correctly determine the distance between two lines in 3D space. They did so primarily by projecting a vector between two points along the joint normal. Very few tried unsuccessfully to determine the distance by minimisation. Since many candidates did not justify their calculations, it is unclear whether they just applied a memorised formula. In part (b), although usually intuitively sound, the provided explanations often lacked in mathematical rigour. A common misconception in part (b.ii) was that basis vectors must be orthogonal. In the final part (b.iii), it was often unclear that the solution is a line parallel to the y-axis.

10. (a)

$$\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = (a-\lambda)\left(\left(\frac{3}{4}-\lambda\right)^2-\frac{1}{16}\right) = (a-\lambda)\left(\frac{1}{2}-\frac{3\lambda}{2}+\lambda^2\right)$$

$$\Rightarrow \lambda_1 = 1\,,\quad \lambda_2 = \frac{1}{2}\,,\quad \lambda_3 = a$$

$$(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{\phi} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{\phi}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}\,,\quad \boldsymbol{\phi}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}\,,\quad \boldsymbol{\phi}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}\,,$$

(b)

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}} \quad \Rightarrow \quad \boldsymbol{A}^n = \left(\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}}\right)\left(\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\mathrm{T}}\right)\cdots = \boldsymbol{U}\boldsymbol{\Lambda}^n\boldsymbol{U}^{\mathrm{T}}$$

Note that $\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} = \boldsymbol{I}$. Hence,

$$\boldsymbol{A}^n = \boldsymbol{U}\boldsymbol{\Lambda}^n\boldsymbol{U}^{\mathrm{T}} = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2^n} & 0\\ 0 & 0 & a^n\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\boldsymbol{A}^n = \boldsymbol{U}\boldsymbol{\Lambda}^n\boldsymbol{U}^{\mathrm{T}} = \begin{bmatrix}\frac{1}{2}+\frac{1}{2^{n+1}} & -\frac{1}{2}+\frac{1}{2^{n+1}} & 0\\ -\frac{1}{2}+\frac{1}{2^{n+1}} & \frac{1}{2}+\frac{1}{2^{n+1}} & 0\\ 0 & 0 & a^n\end{bmatrix}$$

(c) We can distinguish between five cases.

Case $a > 1$:

$$\boldsymbol{A}^n \approx \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & a^n\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & a^n\end{bmatrix}$$

Case $a = 1$:

$$\boldsymbol{A}^n \approx \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

Case: $1/2 < a < 1$:

$$\boldsymbol{A}^n \approx \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & a^n\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & a^n\end{bmatrix}$$

Case $a = 1/2$:

$$\boldsymbol{A}^n \approx \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{2^n} & 0\\ 0 & 0 & \frac{1}{2^n}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}+\frac{1}{2^{n+1}} & -\frac{1}{2}+\frac{1}{2^{n+1}} & 0\\ -\frac{1}{2}+\frac{1}{2^{n+1}} & \frac{1}{2}+\frac{1}{2^{n+1}} & 0\\ 0 & 0 & \frac{1}{2^n}\end{bmatrix}$$

For $a < 1/2$:

$$\boldsymbol{A}^n \approx \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2^{n+1}} & -\frac{1}{2} + \frac{1}{2^{n+1}} & 0 \\ -\frac{1}{2} + \frac{1}{2^{n+1}} & \frac{1}{2} + \frac{1}{2^{n+1}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assessors' remarks: This question was relatively poorly answered. Many candidates did not realise that it is sufficient to consider only a 2x2 matrix to determine the eigenspace of the given 3x3 matrix. However, even failing to see it, one would expect that computing the eigenspace of the given matrix should not be a problem. Having failed to determine the eigenspace in (a) made it, unfortunately, for most candidates, very challenging to come up with sensible answers in (b) and (c). In (b), a common mistake was to use in the spectral decomposition the non-normalised eigenvectors. As expected in (c), some did not realise that they must focus on the eigenvalues and eigenvectors. Others discussed the product of the matrix with an arbitrary vector.

11. (a) Let $s = t - \tau$ so that $ds = d\tau$

$$y(t) = \int_0^t f(\tau)g(t-\tau)d\tau = y(t) = \int_0^t f(t-s)g(s)ds \;\Rightarrow\; y(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

(b) i. Complementary function:

$$y''(t) + 14y'(t) + 49y(t) = 0 \quad \lambda^2 + 14\lambda + 49 = 0 \quad \Rightarrow \quad \lambda = -7$$

General solution:

$$y(t) = \frac{1}{49} + c_1 e^{-7t} + c_2 t e^{-7t}$$

Initial conditions:

$$y(0) = 0\,, \quad y'(0) = 0 \Rightarrow c_1 = \frac{1}{49}\,, \quad c_2 = \frac{1}{7}$$

Step response:

$$y(t) = \frac{1}{49} - \frac{1}{49}e^{-7t} - \frac{t}{7}e^{-7t}$$

ii. Impulse response:

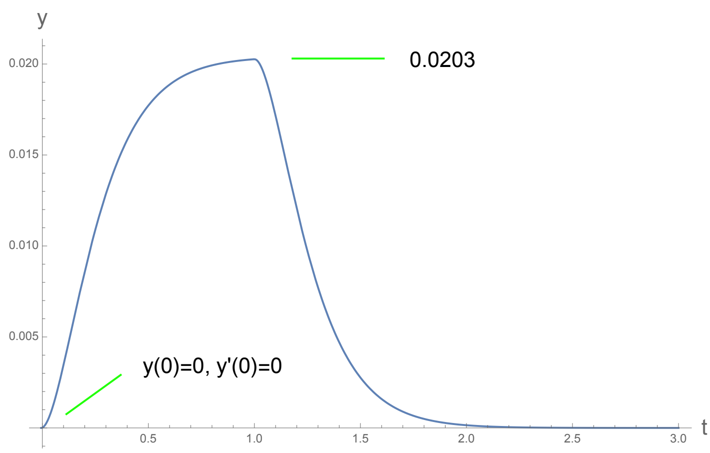
$$\frac{dy(t)}{dt} = te^{-7t}$$

iii. For $t \leq 1$:

$$y(t) = \int_0^t (t-\tau)e^{-7(t-\tau)}d\tau = te^{-7t}\int_0^t e^{7\tau}d\tau - e^{-7t}\int_0^t \tau e^{7\tau}d\tau$$
$$= \frac{t}{7} - \frac{t}{7}e^{-7t} + \frac{1}{49} - \frac{t}{7} - \frac{1}{49}e^{-7t}$$
$$= -\frac{t}{7}e^{-7t} + \frac{1}{49} - \frac{1}{49}e^{-7t}$$

For $t > 1$:

$$y(t) = \int_0^1 (t-\tau)e^{-7(t-\tau)}d\tau = \frac{1}{49}e^{-7t}(-1 - 7t + e^7(-6 + 7t))$$



Assessors' remarks: This question is of the same type as Q5 of example paper 7. The required calculations are straightforward. The easiest way to compute the solution in (b.iii) is to use linearity and construct the solution algebraically, as suggested in the crib of Q5 of paper 7. However, many students unnecessarily used a lengthy method and computed the solution using impulse solution and convolution. Many of them made mistakes in algebra. Furthermore, many students did not group similar terms to obtain a simplified formula.

12. (a)

$$\nabla \boldsymbol{f} = \begin{bmatrix} ye^x + x^2 + 1 \\ \frac{1}{(y+1)(y+2)} + e^x \end{bmatrix}$$

i. Integration of the function yields

$$f = ye^x + \frac{x^3}{3} + x + c_1(y)$$

$$f = \int \left(\frac{1}{1+y} - \frac{1}{2+y} + e^x \right) \mathrm{d}y = \ln(1+y) - \ln(2+y) + ye^x + c_2(x)$$

$$f = ye^x + \frac{x^3}{3} + x + \ln(1+y) - \ln(2+y) + c_3$$

ii.

$$\frac{\partial^2 f}{\partial x \partial y} = e^x \qquad \frac{\partial^2 f}{\partial y \partial x} = e^x$$

Cross derivatives are equal. If not, we would not be able to find f.

iii.

$$\nabla \boldsymbol{c} = \begin{bmatrix} 3e^{3u} \\ 1 \end{bmatrix} \quad \Rightarrow \nabla \boldsymbol{c}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \Rightarrow \frac{\nabla \boldsymbol{c}(0)}{|\nabla \boldsymbol{c}(0)|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\nabla \boldsymbol{f}(x, y) \cdot \frac{\nabla \boldsymbol{c}(0)}{|\nabla \boldsymbol{c}(0)|} = \frac{3}{\sqrt{10}}(ye^x + x^2 + 1) + \frac{1}{\sqrt{10}} \left(\frac{1}{(y+1)(y+2)} + e^x \right)$$

$$\nabla \boldsymbol{f}(0, 0) \cdot \frac{\nabla \boldsymbol{c}(0)}{|\nabla \boldsymbol{c}(0)|} = \frac{9}{2\sqrt{10}}$$

(b) Tangents to the surface

$$\frac{\partial \boldsymbol{r}}{\partial u} = \begin{bmatrix} 2uv \\ 0 \\ v \end{bmatrix} \quad \Rightarrow \frac{\partial \boldsymbol{r}}{\partial u}(1,\, 2) = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\frac{\partial \boldsymbol{r}}{\partial v} = \begin{bmatrix} u^2 \\ 2v \\ u \end{bmatrix} \quad \Rightarrow \frac{\partial \boldsymbol{r}}{\partial v}(1,\, 2) = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Normal

$$\frac{\partial \boldsymbol{r}}{\partial u}(1,\, 2) \times \frac{\partial \boldsymbol{r}}{\partial v}(1,\, 2) = \begin{bmatrix} -8 \\ -2 \\ 16 \end{bmatrix}$$

Assessors' remarks: This question is similar questions to 6(b) and 9 in example paper 10. The required calculations are straightforward. Therefore, it is surprising that many students did not know how to calculate the gradient in a given direction and/or the normal to a surface or couldn't carry out these calculations correctly.

F Cirak
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