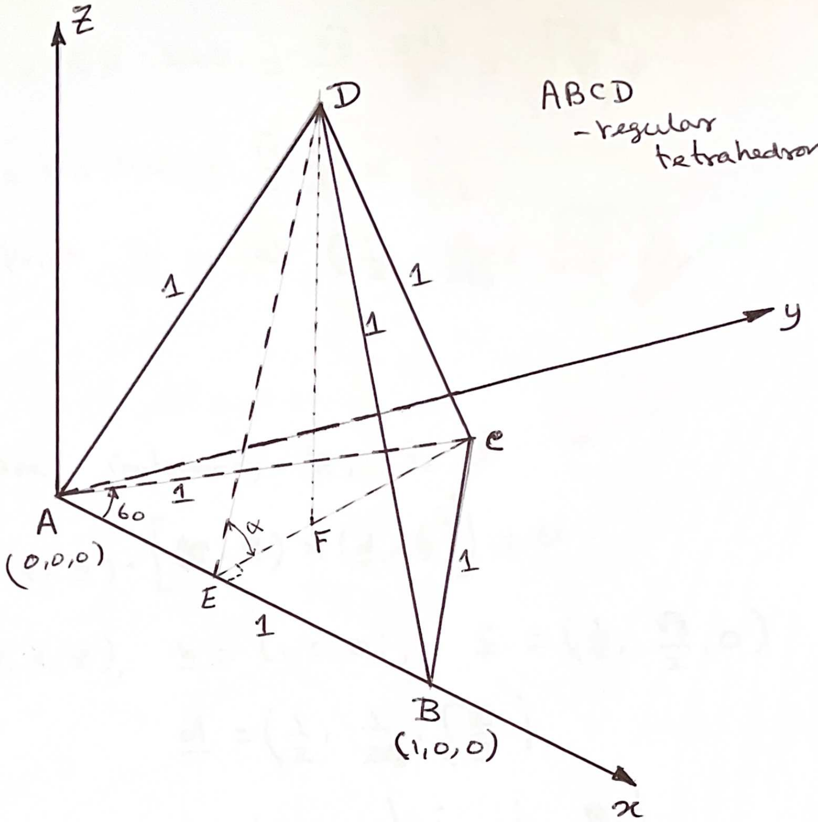
Q9



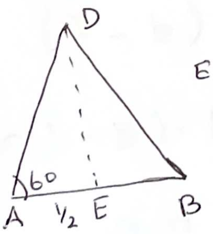
ABCD
– regular tetrahedron

(a) Point C is at

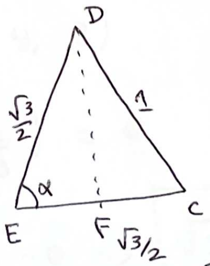
$$x = 1\cos 60, \quad y = 1\sin 60, \quad z = 0$$

$$\Rightarrow C \text{ is } \left(\tfrac{1}{2}, \ \tfrac{\sqrt{3}}{2}, \ 0\right)$$

<u>for Point D</u>: $x = AE = \frac{1}{2}$; $y = EF = ED\cos\alpha$; $z = FD = ED\sin\alpha$



$ED = \frac{\sqrt{3}}{2}$



$EC = \frac{\sqrt{3}}{2} = 1\sin 60$ (See the above diagram)

Cosine rule $\Rightarrow$ $1^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 - 2\,\frac{\sqrt{3}}{2}\,\frac{\sqrt{3}}{2}\cos\alpha$

$$\Rightarrow \cos\alpha = \frac{1}{3} \Rightarrow \sin\alpha = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

$\therefore \; z = ED \sin\alpha = \frac{\sqrt{3}}{2} \frac{2\sqrt{2}}{3} = \sqrt{\frac{2}{3}}$

$y = ED \cos\alpha = \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \frac{1}{2\sqrt{3}}$

So the point D is at $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$

(b)

(i) plane containing $\underline{b}$, $\underline{c}$, $\underline{d}$ is

$$(\underline{r} - \underline{b}) \cdot \left[(\underline{c} - \underline{b}) \times (\underline{d} - \underline{b})\right] = 0$$

$\underline{r} = (x, y, z)$, $\underline{b} = (1, 0, 0)$, $\underline{c} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$

$\underline{d} = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right)$

$$(\underline{c} - \underline{b}) \times (\underline{d} - \underline{b}) = \begin{vmatrix} i & j & k \\ -1/2 & \frac{\sqrt{3}}{2} & 0 \\ -1/2 & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{2\sqrt{3}}\right)$$

$(\underline{r} - \underline{b}) = (x-1, y, z)$

$\Rightarrow (\underline{r} - \underline{b}) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{2\sqrt{3}}\right) = 0$

$$\frac{(x-1)}{\sqrt{2}} + \frac{y}{\sqrt{6}} + \frac{z}{2\sqrt{3}} = 0 \Rightarrow \boxed{\sqrt{3}x + y + \frac{z}{\sqrt{2}} = \sqrt{3}}$$

(ii) Normal to BCD plane is

Parallel to $\left(\sqrt{3}, 1, \frac{1}{\sqrt{2}}\right)$ as suggested by the plane equation.

$\Rightarrow$ line through $A = (0,0,0)$ is

$$\boxed{\frac{x}{\sqrt{3}} = y = \sqrt{2}\, z}$$

(iii) line through D & $\perp$r to ABC plane

is DF in the figure.

$\Rightarrow$ z line going through $x = \frac{1}{2}$; $y = \frac{1}{2\sqrt{3}}$
(from (a))

(c) $x = 1/2$; $y = \frac{1}{2\sqrt{3}}$ on b(ii) line

$\Rightarrow$ $z = \frac{x}{\sqrt{6}}$ or $z = \frac{y}{\sqrt{2}}$

$\Rightarrow$ $\boxed{z = \frac{1}{2\sqrt{6}}}$ The point of intersection is $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{6}}\right)$.

This point is the centre of mass for the given regular tedrahedron.

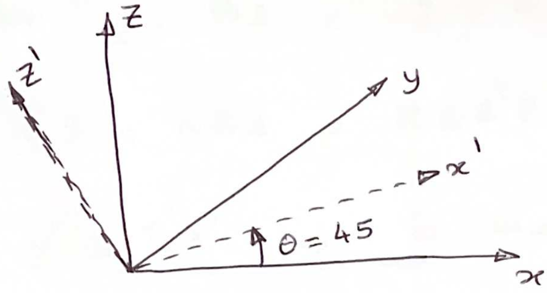
(d) Volume $= \frac{\cancel{a}^{1\,3}}{6\sqrt{2}} = \underline{\underline{\frac{1}{6\sqrt{2}}}}$

$\left[\frac{1}{6}\text{th of volume of a parallelogram formed by } \underline{b}, \underline{c}, \underline{d} \Rightarrow \frac{1}{6}\, \underline{b} \cdot (\underline{c} \times \underline{d}) \right]$

Q10

(a)

(i)



$$x' = x\cos\theta - z\sin\theta$$

$$y' = y$$

$$z' = x\sin\theta + z\cos\theta$$

$\theta = 45^\circ$

$$\Rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos 45 & 0 & -\sin 45 \\ 0 & 1 & 0 \\ \sin 45 & 0 & \cos 45 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$V' = RV \Rightarrow \boxed{R = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}}$$

(ii) Consider $\underline{y} = B\underline{x}$, $\underline{y}' = B'\underline{x}'$

$$R\underline{y} = RB\underline{x} = RBR^TR\underline{x}$$

$\Rightarrow$ $\underline{y}' = B'\underline{x}'$ $\Rightarrow$ B becomes $B' = RBR^T$

(b) $\left.\begin{matrix}\lambda_1, & \lambda_2, & \lambda_3 \\ e_1, & e_2, & e_3\end{matrix}\right\}$ Eigen values & vectors of A.

$\Rightarrow A^n$ will have $\left.\begin{matrix}\lambda_1^n, & \lambda_2^n & \lambda_3^n \\ e_1, & e_2 & e_3\end{matrix}\right\}$ as its eigen values & vectors.

(c) $A = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}$

$$\begin{vmatrix} 10-\lambda & 0 & 0 \\ 0 & 3-\lambda & -2 \\ 0 & -2 & 3-\lambda \end{vmatrix} = (10-\lambda)\left((3-\lambda)^2 - 4\right) = 0$$

$\Rightarrow \lambda_1 = 10$; $\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda_3 = 1,\ \lambda_2 = 5.$

$\Rightarrow$ eigen values are $10, 5, 1$

$A\underline{x} = \lambda \underline{x} \Rightarrow$ for $\lambda = 10$

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 10\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow (1, 0, 0) = e_1$ for $\lambda = 10$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

for $\lambda = 5$

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\Rightarrow \quad x = 0$

$$\left. \begin{matrix} 2y + 2z = 0 \\ 2y + 2z = 0 \end{matrix} \right\} \Rightarrow y = -z; \qquad \therefore e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

for $\lambda = 1$

$$\begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x = 0 \\ y = z \end{matrix}$$

$$\therefore e_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

(ii) $\underline{x} = (1, 2, 1)^T$

writing this on the basis of e_1, e_2, e_3

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

for A^{10} eigen vectors remain the same and eigen values are 10^{10} 5^{10} 1

$$\therefore A^{10}\underline{x} = \begin{pmatrix} 10^{10} \\ \frac{5^{10}}{2} + 3/2 \\ -\frac{5^{10}}{2} + 3/2 \end{pmatrix} \approx 10^{10} \begin{pmatrix} 1 \\ 1/2^{11} \\ -1/2^{11} \end{pmatrix} \approx 10^{10} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This can also be shown using $A^{10}\underline{x} = B\underline{x} = (V \Lambda V^{-1})\underline{x}$

Q11

$$\beta \frac{dy}{dt} + y = f(t) \qquad y(t) = 0 \text{ for } t < 0$$

(a) C.F. is $y(t) = A e^{-t/\beta}$

P.I. is $y = 1$

$\Rightarrow$ General Solution is $y(t) = 1 + A e^{-t/\beta}$

$y(t)$ must be continuous @ $t = 0$

$\Rightarrow y(0^-) = y(0^+) = 0$

$\Rightarrow A = -1$

$\therefore$ The step response is $\boxed{y(t) = 1 - e^{-t/\beta}}$

(b) Impulse response is $\frac{d}{dt}$ (Step response)

$\Rightarrow$ Impulse response is $\boxed{\frac{1}{\beta} e^{-t/\beta}} = g(t)$

(c) $f(t)$ can be approximated as impulse chain

$$\Rightarrow f(t) = \sum_{k=0}^{\infty} f(kT)\, \delta(t - kT)\, T$$

this give a response as (for linear system as it is the case here)

$$y(t) = \sum_{k=0}^{\infty} f(kT)\, T\, g(t - kT)$$

as $T \to 0$ $\quad kT \to \tau$ & $T \to d\tau$

$$\Rightarrow y(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau$$

We have assumed that f(t) = 0 for t < 0.

(d) $$y(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \frac{1}{\alpha\beta} \int_0^t e^{\tau/\alpha}\, e^{-(t-\tau)/\beta}\, d\tau$$

$$= \frac{e^{-t/\beta}}{\alpha\beta} \int_0^t e^{-\tau\varphi}\, d\tau \qquad \varphi = \frac{\beta-\alpha}{\alpha\beta} = \frac{1}{\alpha} - \frac{1}{\beta}$$

$$= \frac{e^{-t/\beta}}{\alpha\beta} \left[\frac{e^{-\tau\varphi}}{-\varphi} \right]_0^t$$

$$= \frac{e^{-t/\beta}}{\beta-\alpha} \left[1 - e^{-t\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)} \right]$$

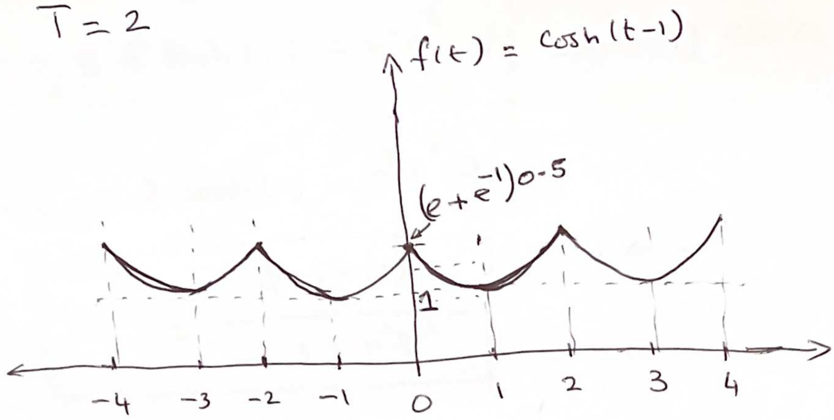
$$\Rightarrow \boxed{y(t) = \frac{1}{\beta-\alpha} \left[e^{-t/\beta} - e^{-t/\alpha} \right]}$$

$$\alpha \to 0 \Rightarrow y(t) \simeq \frac{e^{-t/\beta}}{\beta} \to \text{impulse response}$$

Q12

$T = 2$

(a)



(b) This function is even function

$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$$

$$a_n = \frac{2}{T}\int_{-1}^{1} f(t) \cos n\pi t \, dt$$

$$= 2\int_{0}^{1} \cosh(t-1) \cos(n\pi t)\, dt$$

$$= 2\int_{0}^{1} \cos n\pi t \; d[\sinh(t-1)]$$

$\int u\, dv = uv - \int v\, du$

$$= 2 \sinh(t-1) \cos n\pi t \Big|_0^1 + 2n\pi \int_0^1 \sinh(t-1) \sin n\pi t \, dt$$

$$= 2 \sinh(1) + 2n\pi \int_0^1 \sin n\pi t \; d[\cosh(t-1)]$$

$$= 2\sinh(1) + 2n\pi \left\{ \cosh(t-1) \sin n\pi t \Big|_0^1 - n\pi \int_0^1 \cosh(t-1) \cos n\pi t \, dt \right\}$$

$$= 2\sinh(1) - n^2\pi^2\, 2\int_0^1 \cosh(t-1)\cos n\pi t\, dt$$

$$a_n = 2\sinh(1) - n^2\pi^2 a_n$$

$$\Rightarrow \boxed{a_n = \frac{2\sinh(1)}{1+n^2\pi^2}}$$

$$\Rightarrow a_0 = 2\sinh(1)$$

$$\therefore f(t) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1+n^2\pi^2}$$

$$\Rightarrow \boxed{f(t) = \sinh(1)\left[1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1+n^2\pi^2}\right]}$$

as required.

(c) for $t=0$

$$\cosh(-1) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}$$

$$\frac{e+e^{-1}}{2} = \frac{e-e^{-1}}{2} + (e-e^{-1})\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}$$

Simplifying & rearranging gives

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2+1} = \frac{1}{e^2-1}}$$