Version NS/1

EGT0
ENGINEERING TRIPOS PART IA

Tuesday 13 June 2023 9 to 12.10

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Section A: multiple choice supplementary booklet
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Questions 1–8: see multiple choice supplementary booklet.

SECTION B

9 (**long**) The regular tetrahedron $ABCD$ has its vertex A at the origin, vertex B at $(1, 0, 0)$, vertex C in the plane $z = 0$ with $y > 0$, and vertex D in $z > 0$.

(a) Show that the coordinates of C and D are

$$\left(\frac{1}{2}\,,\ \frac{\sqrt{3}}{2}\,,\ 0\right) \quad \text{and} \quad \left(\frac{1}{2}\,,\ \frac{1}{2\sqrt{3}}\,,\ \sqrt{\frac{2}{3}}\right)$$

respectively. [10]

(b) Determine the equation of:

(i) the plane containing BCD; [7]

(ii) the line through A perpendicular to the plane containing BCD; [3]

(iii) the line through D perpendicular to the plane containing ABC. [2]

(c) Find the intersection of the lines described in b(ii) and b(iii) above. What is the physical significance of this point? [5]

(d) Find the volume of this regular tetrahedron. [3]

10 (**long**)

(a) The $x'y'z'$ axes are obtained by rotating the xyz axes by an angle 45° anticlockwise about the y axis.

(i) Determine the matrix R such that the vector $\mathbf{v}$ in the xyz system takes the form $R\mathbf{v}$ in the $x'y'z'$ system. [6]

(ii) Consider a matrix B in the xyz system. Explain why B becomes $B' = RBR^T$ in the $x'y'z'$ system. [5]

(b) The 3×3 matrix A has eigenvalues λ_1, λ_2, λ_3 and corresponding eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Determine the eigenvalues and corresponding eigenvectors of A^n, where n is a positive integer. [5]

(c) Consider the matrix

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}$$

(i) Calculate λ_1, λ_2, λ_3 and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, the eigenvalues and eigenvectors of A. [8]

(ii) Calculate $A^{10}\mathbf{x}$ for $\mathbf{x} = (1, 2, 1)^T$. [6]

SECTION C

11 (**long**) Consider a linear system with input $f(t)$ governed by the equation

$$\beta \frac{dy}{dt} + y = f(t)$$

where β is a positive constant.

(a) Find the step response of the system without using Laplace transforms, taking $y(t) = 0$ for $t < 0$. [5]

(b) Find the impulse response, $g(t)$, of the system. [4]

(c) Explain clearly why the response to a forcing function $f(t)$ can be written as

$$y(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

State any assumptions you make. [6]

(d) Find the system response for

$$f(t) = \begin{cases} 0 & t < 0 \\ \dfrac{1}{\alpha} \exp\left(-t/\alpha\right) & t \geq 0 \end{cases}$$

where $\alpha \neq \beta$ is a positive constant. What is the significance of the limit $\alpha \to 0$? Comment briefly on your result. [15]

12 (**long**) An even function $f(t)$ is periodic with period $T = 2$ and $f(t) = \cosh(t - 1)$ for $0 \leq t \leq 1$.

(a) Sketch $f(t)$ in the range $-4 \leq t \leq 4$. [5]

(b) Show that

$$f(t) = \sinh(1)\left[1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\pi t)}{1 + n^2\pi^2}\right]$$

is a Fourier series representation of $f(t)$. [15]

(c) Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2\pi^2} = \frac{1}{e^2 - 1}$$

using the solution in (b). [10]

END OF PAPER