Paper 4: Mathematical Methods

Solutions to 2025 Sections B and C

9. **Complex numbers**

(a) (i)

$$|z-4| = |z-2-2i|$$
$$|x-4+iy|^2 = |x-2+i(y-2)|^2$$
$$x^2-8x+16+y^2 = x^2-4x+4+y^2-4y+4$$

$$\Rightarrow y = x-2$$

A line with slope 1 and intercept -2. [3]

(ii)

$$|z-4| = \sqrt{2}|z-2-2i|$$
$$|x-4+iy|^2 = 2|x-2+i(y-2)|^2$$
$$x^2-8x+16+y^2 = 2x^2-8x+8+2y^2-8y+8$$
$$x^2+y^2-8y = 0$$

$$\Rightarrow x^2+(y-4)^2 = 16$$

A circle with radius 4 centred at $(0,\,4)$. [4]

(b)

$$\int \exp(ax)(\cos(bx)+i\sin(bx))\,dx = \int \exp(ax+ibx)dx$$
$$= \frac{\exp(ax+ibx)}{a+ib}+c$$
$$= \frac{(a-ib)\exp(ax+ibx)}{a^2+b^2}+c$$
$$= \frac{\exp(ax)(a-ib)\exp(ibx)}{a^2+b^2}+c$$

Taking only the real terms on both sides yields

$$\int \exp(ax)\cos(bx) = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx)+b\sin(bx)\right)+c$$ [6]

(c) (i)

$$z = 2+2i$$

Expressed in the polar form as

$$z = |z|e^{i\theta} \text{ with } |z| = 2\sqrt{2},\ \theta = \tan^{-1}1$$

Roots are given by

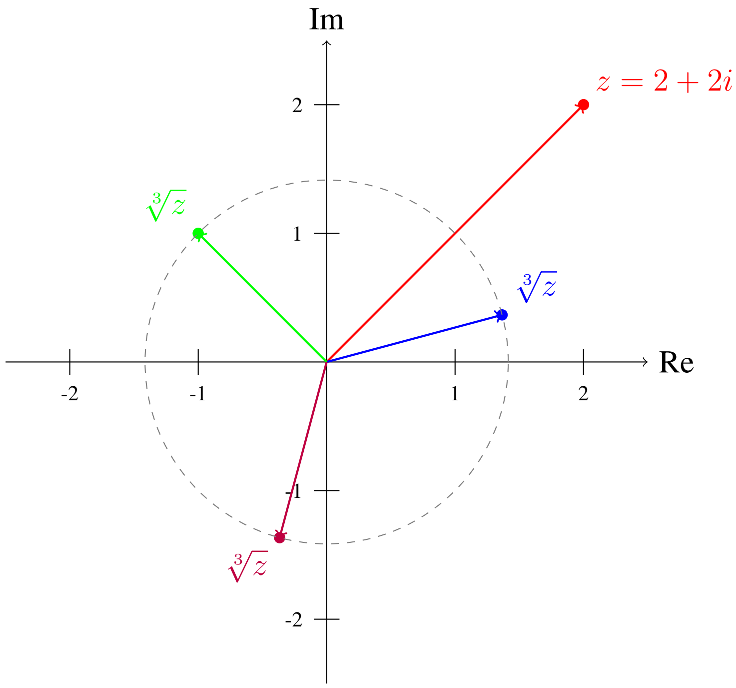
$$z^{1/3} = |z|^{1/3}e^{i(\theta+2n\pi)/3}$$

where $n \in \{0, 1, 2\}$.

$$z^{1/3} = |z|^{1/3} e^{j\theta/3},$$
$$z^{1/3} = |z|^{1/3} e^{j(\theta+2\pi)/3},$$
$$z^{1/3} = |z|^{1/3} e^{j(\theta+4\pi)/3}.$$

More explicitly,

$$z^{1/3} = \sqrt{2} e^{j\pi/12} = 1.3660 + 0.3660j,$$
$$z^{1/3} = \sqrt{2} e^{j3\pi/4} = -1 + j,$$
$$z^{1/3} = \sqrt{2} e^{j17\pi/12} = -0.3660 - 1.3660j.$$



[8]

(ii) Substitute

$$y = z^3$$

Hence,

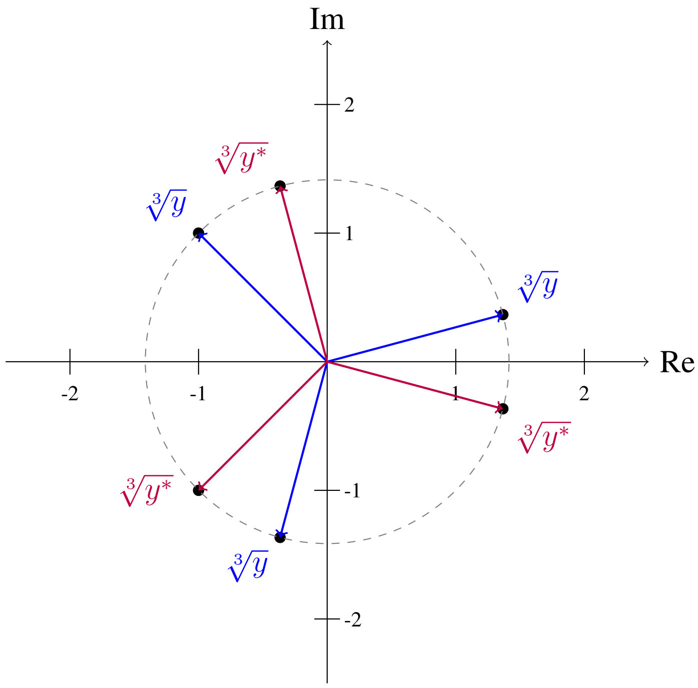
$$y^2 - 4y + 8 = 0$$
$$y_{1,2} = \frac{4 \pm \sqrt{4^2 - 4 \cdot 8}}{2} = 2 \pm 2i$$

Hence,

$$z_1 = y^{1/3}, \quad z_2 = (y^*)^{1/3}$$

Roots of $y^{1/3}$ as in part (i) and the $(y^*)^{1/3}$ are obtained by reflecting $y^{1/3}$ about the horizontal real axis, i.e.,

$$(y^*)^{1/3} = 1.3660 - 0.3660j,$$
$$(y^*)^{1/3} = -1 - j.$$
$$(y^*)^{1/3} = -0.3660 + 1.366j$$



[9]

Assessor's remarks: A very well answered question with a high average mark, suggesting that it may have been too easy. In (a), there were only a few problems in correctly identifying the loci and properly describing their geometry. The two most common mistakes in (b) were to express the cosine term as the sum of two exponentials, leading to lengthy, error-prone computations, and forgetting the constant of integration. In (c), most candidates were able to determine the roots. However, few candidates recognised that the new roots in (ii) were the complex conjugates of the ones in (i). Although the question asked for plots of the roots in Argand diagrams, most candidates provided only poor sketches.

10. **Eigenvalues and eigenvectors, series expansions**

(a)

$$\det(\mathbf{A}-\lambda\mathbf{I}) = (a-\lambda)\left((1-\lambda)^2 - a^2\right) = (a-\lambda)\left(1-2\lambda+\lambda^2-a^2\right)$$

$$\Rightarrow \lambda_1 = (1-a)\,, \quad \lambda_2 = a\,, \quad \lambda_3 = (1+a)$$

$$(\mathbf{A}-\lambda\mathbf{I})\phi = \mathbf{0}$$

$$\Rightarrow \phi_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \phi_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \phi_3 = \begin{bmatrix}1\\1\\0\end{bmatrix},$$

[10]

(b) (i)

$$\mathbf{A} = \mathbf{U\Lambda U}^{\mathrm{T}} \quad \Rightarrow \quad \mathbf{A}^n = \left(\mathbf{U\Lambda U}^{\mathrm{T}}\right)\left(\mathbf{U\Lambda U}^{\mathrm{T}}\right)\cdots = \mathbf{U\Lambda}^n\mathbf{U}^{\mathrm{T}}$$

Note that $\mathbf{UU}^{\mathrm{T}} = \mathbf{I}$. Hence,

$$\mathbf{A}^n = \mathbf{U\Lambda}^n\mathbf{U}^{\mathrm{T}} = \frac{1}{2}\begin{bmatrix}-1 & 0 & 1\\ 1 & 0 & 1\\ 0 & \sqrt{2} & 0\end{bmatrix}\begin{bmatrix}\lambda_1^n & 0 & 0\\ 0 & \lambda_2^n & 0\\ 0 & 0 & \lambda_3^n\end{bmatrix}\begin{bmatrix}-1 & 1 & 0\\ 0 & 0 & \sqrt{2}\\ 1 & 1 & 0\end{bmatrix}$$

$$\mathbf{A}^n = \mathbf{U}\boldsymbol{\Lambda}^n\mathbf{U}^{\mathrm{T}} = \frac{1}{2}\begin{bmatrix} \lambda_1^n + \lambda_3^n & -\lambda_1^n + \lambda_3^n & 0 \\ -\lambda_1^n + \lambda_3^n & \lambda_1^n + \lambda_3^n & 0 \\ 0 & 0 & 2\lambda_2^n \end{bmatrix}$$ [8]

(ii)

$$\exp(\mathbf{A}) = \mathbf{U}\mathbf{I}\mathbf{U}^{\mathrm{T}} + \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}} + \frac{1}{2!}\mathbf{U}\boldsymbol{\Lambda}^2\mathbf{U}^{\mathrm{T}} + \frac{1}{3!}\mathbf{U}\boldsymbol{\Lambda}^3\mathbf{U}^{\mathrm{T}} + \cdots$$

$$\exp(\mathbf{A}) = \mathbf{U}\left(\mathbf{I} + \boldsymbol{\Lambda} + \frac{1}{2!}\boldsymbol{\Lambda}^2 + \frac{1}{3!}\boldsymbol{\Lambda}^3 + \cdots\right)\mathbf{U}^{\mathrm{T}}$$

The bracketed expression is a diagonal matrix, and each of its entries represents the series expansion of a standard exponential function. Hence,

$$\exp(\mathbf{A}) = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\exp(\mathbf{A}) = \frac{1}{2}\begin{bmatrix} e^{\lambda_1} + e^{\lambda_3} & -e^{\lambda_1} + e^{\lambda_3} & 0 \\ -e^{\lambda_1} + e^{\lambda_3} & e^{\lambda_1} + e^{\lambda_3} & 0 \\ 0 & 0 & 2e^{\lambda_2} \end{bmatrix}$$

$$\exp(\mathbf{A}) = \begin{bmatrix} e\cosh a & e\sinh a & 0 \\ e\sinh a & e\cosh a & 0 \\ 0 & 0 & e^a \end{bmatrix}$$ [9]

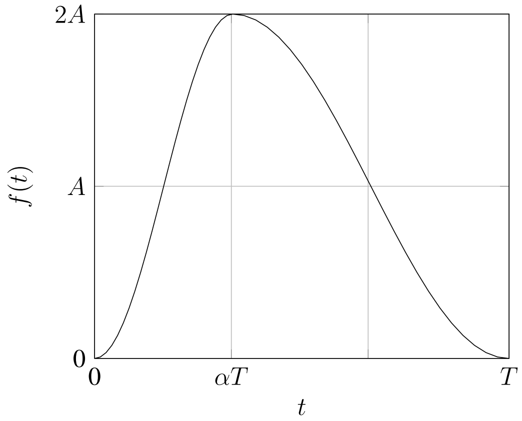
(iii) For small a the first order expansion of $\cosh$, $\sinh$ and e yields

$$\exp(\mathbf{A}) \approx \begin{bmatrix} e & ea & 0 \\ ea & e & 0 \\ 0 & 0 & 1 + a \end{bmatrix}$$ [3]

Assessor's remarks: Part (a) of this question was answered very well, with only a few candidates unable to determine the correct eigenvalues and eigenvectors. Many candidates failed to see by inspection that one only needs to consider a two-by-two eigenvalue problem. Part (b) was less well answered. Common mistakes included not normalising the rows in $\mathbf{U}$ and forgetting that it is an orthogonal matrix, which can be trivially inverted. The vast majority of candidates did not recognise in (ii) that one can simply apply the exponential function to the diagonal entries of the eigenvalue matrix. There were only a handful of correct results in (iii) since it requires the correct solution of (ii).

11. **Fourier series**

(a) By examining the values of $f(t)$, at 0, αT and T, we sketch the function as



[5]

(b) The parameter d is obtained by integrating the function over the interval

$$d = \frac{1}{T}\int_0^T f(t)dt.$$

From the sketch, the function is half a period of a cosine (i.e. from 0 to π) on the interval $[0, \alpha T]$, followed by a second half period of a cosine (i.e., from π to 2π) on the interval $[\alpha T, T]$, plus a constant A. A cosine integrates to zero over each half period, so the only non-zero component of the integral is the constant, which integrates to AT, hence $d = A$.

[5]

(c) When $\alpha = \frac{1}{2}$, the expressions for $t \leq \alpha T$ and for $t \geq \alpha T$ coincide over $[0, T]$, hence

$$f(t) = A - A\cos\left(\pi\frac{t}{\alpha T}\right)$$

which is already in Fourier series form. [5]

(d) $f(t)$ is continuous because $f_L(\alpha T) = f_R(\alpha T) = 2A$. We compute the derivatives

$$f_L'(t) = \frac{A\pi}{\alpha T}\sin\left(\pi\frac{t}{\alpha T}\right) \quad \text{and} \quad f_R'(t) = -\frac{A\pi}{(1-\alpha)T}\sin\left(\pi\frac{t-\alpha T}{T-\alpha T}\right)$$

This continuous at $t = \alpha T$ since $f_R'(\alpha T) = f_L'(\alpha T) = 0$. We now compute

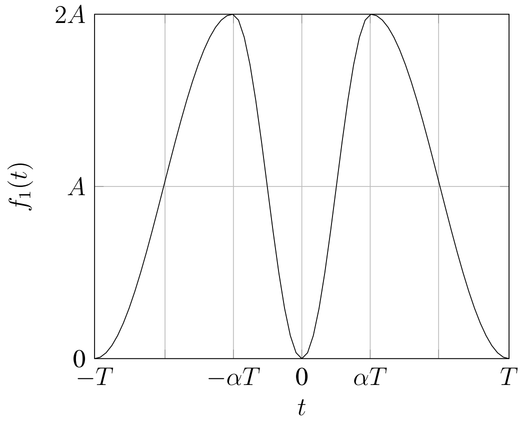
$$f_L''(t) = \frac{A\pi^2}{(\alpha T)^2}\cos\left(\pi\frac{t}{\alpha T}\right) \quad \text{and} \quad f_R''(t) = -\frac{A\pi^2}{(1-\alpha)^2T^2}\cos\left(\pi\frac{t-\alpha T}{T-\alpha T}\right)$$

and note that

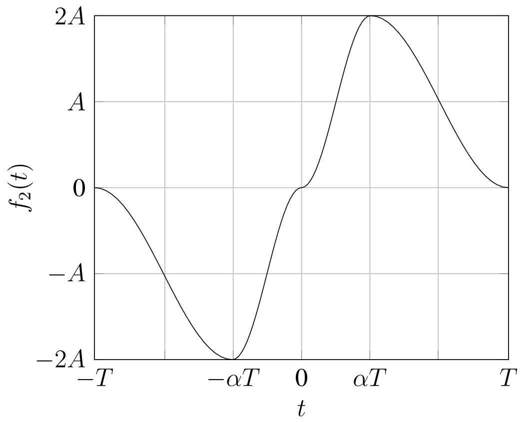
$$f_L''(\alpha T) = -\frac{A\pi^2}{(\alpha T)^2} \neq f_R''(\alpha T) = -\frac{A\pi^2}{(1-\alpha)^2T^2} \quad \text{if } \alpha \neq \frac{1}{2}.$$

Hence, there is a discontinuity in $f''(t)$, resulting in a pulse function in $f'''(t)$. Hence, the Fourier coefficients of $f'''(t)$ don't decay, those of $f''(t)$ decay with $1/n$, those of $f'(t)$ decay with $1/n^2$, and those of $f(t)$ decay with $1/n^3$. [8]

(e) $f_1(t)$ has a constant term and a sum of terms multiplying only cosines and no sines. Hence it is a an even function. If it is equal to $f(t)$ for $0 \leq t \leq T$, it must be the even reflection of $f(t)$ at negative times and its period must be $2T$. Drawing $f_1(t)$ from $-T$ to T we obtain



$f_2(t)$ on the other hand has only terms multiplying sines of t, no constant and no cosines. It is hence an odd function. If it equal to $f(t)$ for $0 \le t \le T$, it must be the odd reflection of $f(t)$ at negative times and its period is also $2T$. Drawing $f_2(t)$ from $-T$ to T we obtain



[7]

Note: the question did not require students to compute the Fourier coefficients (it's a long and tedious calculation) but if anyone wants to have a go and check their work, here are the full expressions (numerically verified to be correct):

$$f_1(t) = A + \frac{A}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\alpha}{1-n(1-\alpha)} - \frac{1-\alpha}{1+n(1-\alpha)} + \frac{\alpha}{1+n\alpha} - \frac{\alpha}{1-n\alpha}\right)\sin(\pi n\alpha)\cos\left(\frac{\pi n t}{T}\right)$$

$$f_2(t) = \frac{A}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{n}\left(1-(-1)^n\right) - \left(\frac{\alpha}{1+n\alpha} - \frac{\alpha}{1-n\alpha}\right)(1+\cos\pi n\alpha) + \left(\frac{1-\alpha}{1+n(1-\alpha)} - \frac{1-\alpha}{1-n(1-\alpha)}\right)\left((-1)^n + \cos\pi n\alpha\right)\right]\sin\left(\frac{\pi n t}{T}\right)$$

Note that it may at fist glance appear as if the coefficients decay with $1/n$ since the expressions have only n's in the denominator, but the expressions are a "partial fraction" expansion: if you combine them into a rational expression, coefficients in the numerator cancel and you indeed end up with an expression decaying as $O(n^{-3})$ as stated in part (d).

Assessor's remarks: This question examined candidates' understanding of Fourier series. The different parts of the question probed the following knowledge:

a) Plot the graph of a function consisting of two half raised cosines of different frequencies. Many candidates did this perfectly, but some struggled.

b) Determine the constant of the Fourier series, which evidently is simply the bias that raises the cosines. Candidates had been specifically asked to justify this from their graph and some did, but many used the definition of Fourier series from the data book to derive the constant term algebraically.

c) Compute the Fourier series for the case when the two frequencies are the same. In this case the function is simply a constant plus a cosine, which is already a Fourier series, and many candidates saw this immediately but some launched into long calculations using the definition of Fourier series and only a few of those who did this ended up computing the correct two-term Fourier series.

d) Explain why the Fourier coefficients decay as $O(n^{-3})$ when the two frequencies are different. This is a direct application of material taught in lectures and practiced in an examples paper, and required candidates to show that the second derivative was discontinuous at the point where the two half cosines meet. Some candidates did this, some only said that the second derivative had to be discontinuous without actually deriving it, some had a vague idea that continuity of derivatives had something to do with this, and a few candidates had no clue.

e) Two Fourier series expansions were given, one with only a constant and cosines, and one with only sines. Candidates were told that these two expansions were equal to the function given in the interval [0,T] and asked to draw the two functions over one of their respective periods. This was closely based on a question in an examples paper (where students work out a Fourier series of a guitar string using only sines) and many candidates understood that the period had to be 2T, where the first function was even and the second odd, but some candidates missed this completely and some drew the functions over 4T or just re-drew their answer from part (a) twice from 0 to T.

12. **Laplace transforms**

(a) We transform the system of differential equations into the Laplace domain

$$\begin{cases} s\overline{y} - y(0) + 4\overline{x} = \overline{u} \\ s\overline{x} - x(0) - \overline{y} = \overline{w} \end{cases}$$

Taking the first equation plus s times the second,

$$4\overline{x} + s^2\overline{x} - y(0) - sx(0) = \overline{u} + s\overline{w}$$

and hence

$$\overline{x} = \frac{\overline{u} + s\overline{w}}{s^2+4} + \frac{y(0) + sx(0)}{s^2+4}$$ [6]

(b) The Laplace transform of $w(t) = H(t)$ is $\overline{w}(s) = s^{-1}$. Hence using the previous result with $x(0) = y(0) = 0$,

$$\overline{x}(s) = \frac{1}{s^2+4}$$

which, using the table in the data book, has the inverse transform

$$\begin{cases} x(t) = \frac{1}{2}\sin(2t) \text{ for } t \geq 0, \\ x(t) = 0 \text{ for } t < 0. \end{cases}$$

Going back to the system of equations

$$\overline{y} = s\overline{x} - \overline{w} = \frac{s}{s^2+4} - \frac{1}{s}$$

again using the inverse transform in the data book, yields

$$\begin{cases} y(t) = \cos(2t) - 1 \text{ for } t \geq 0 \\ y(t) = 0 \text{ for } t < 0. \end{cases}$$ [8]

(c) We use the convolution property, noting that

$$\overline{g}_1(s) = \frac{s}{s^2+4} \text{ and } \overline{g}_2(s) = \frac{1}{s^2}$$

the Laplace transform of the result is

$$\overline{g}_1(s) \cdot \overline{g}_2(s) = \frac{s}{s^2(s^2+4)} = \frac{s}{4}\left(\frac{1}{s^2} - \frac{1}{s^2+4}\right) = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right).$$

Using the Laplace table in the data book,

$$\int_0^t g_1(\tau)g_2(t-\tau)d\tau = \begin{cases} \frac{1}{4}\left(1 - \cos(2t)\right) \text{ for } t \geq 0, \\ 0 \text{ for } t < 0. \end{cases}$$ [8]

(d) We can either derive this from first principles, or notice that the step response for the $x(t)$ output that we computed in (b) is $\frac{1}{2}\sin(2t)$, and hence its derivative $\cos(2t)$ is the impulse response. Hence, we can obtain the output by convolving $\cos(2t)$ with the input $-2t$. We've already computed the convolution of $\cos(2t)$ with t in (c), and hence the output should be -2 times the answer in (c),

$$\begin{cases} x(t) = \frac{1}{2}\left(\cos(2t) - 1\right) \text{ for } t \geq 0 \\ x(t) = 0 \text{ for } t < 0. \end{cases}$$ [8]

Assessor's remarks: This question probed students' understanding of Laplace transforms which are taught in the very last weeks of term. It was thought that students might struggle with such a question because the material is so recent, but on the contrary, students did very well. The question asked candidates to solve a system of linear differential equations in the Laplace domain. The first "show that" part was almost universally correctly answered. There was some variability in the answers to the remaining questions. Question (b) asked candidates to compute the solutions for a given input using Laplace transforms and not otherwise, but many candidates fell back to solving a differential equation in the time domain for the second half of the question. Questions (c) and (d) essentially asked candidates to compute convolutions using Laplace transforms, once explicitly in (c) and then in (d), the convolution was implicit in that candidates were required to convolve an input signal with a system impulse response. Most candidates had no trouble at all with both of these questions, but there was a small number of candidates who had no trouble with question (d), where you had to "multiply an input signal by a transfer function", but struggled with question (c) when asked to perform a convolution in the Laplace domain. This shows that they have internalised the concept of a multiplicative transfer function in the Laplace domain but not truly understood the link to convolutions in the time domain.

Fehmi Cirak and Jossy Sayir
June 2025