Version JS/FC/1

EGT0
ENGINEERING TRIPOS PART IA

Tuesday 17 June 2025 9 to 12.10

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Section A: multiple choice supplementary booklet
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Questions 1–8: see multiple choice supplementary booklet.

SECTION B

9 (**long**)

(a) The complex variable z satisfies the equation

$$|z - 4| = \lambda |z - z_0|$$

where $z_0 = 2 + 2i$.

(i) Find the equation of the locus of z for $\lambda = 1$ and describe its geometry. [3]

(ii) Find the equation of the locus of z for $\lambda = \sqrt{2}$ and describe its geometry. [4]

(b) Integrate the expression

$$\int e^{ax} \cos(bx)\, dx$$

by expressing the integrand as the real part of $e^{ax} e^{ibx}$. [6]

(c) (i) Find the roots of $z^{1/3}$ where

$$z = 2 + 2i$$

and plot all of these roots in the complex plane on an Argand diagram. [8]

(ii) Solve the equation

$$z^6 - 4z^3 + 8 = 0$$

and plot the position of all its roots in the complex plane on an Argand diagram. [9]

(TURN OVER

10 (**long**) The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & a & 0 \\ a & 1 & 0 \\ 0 & 0 & a \end{bmatrix}$$

where a is a positive constant.

(a) Find the eigenvalues and eigenvectors of $\mathbf{A}$. [10]

(b) The matrix exponential is defined as

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}$$

The matrix $\mathbf{A}$ can be expressed as $\mathbf{A} = \mathbf{U\Lambda U}^{-1}$, where $\mathbf{U}$ is a proper orthogonal matrix and $\mathbf{\Lambda}$ is a diagonal matrix.

(i) Find the matrix $\mathbf{A}^n$ for an arbitrary non-negative integer n. [8]

(ii) Find the matrix $e^{\mathbf{A}}$, giving your answer as an explicit expression in terms of e and a. [9]

(iii) If a is small, find an approximation to $e^{\mathbf{A}}$. [3]

SECTION C

11 (**long**) The expected excess sunspot count over a solar cycle of duration T is given by the function

$$f(t) = \begin{cases} A\left(1 - \cos\left(\frac{\pi t}{\alpha T}\right)\right) & \text{for } 0 \le t \le \alpha T \\ A\left(1 + \cos\left(\frac{\pi(t - \alpha T)}{T - \alpha T}\right)\right) & \text{for } \alpha T \le t \le T \end{cases}$$

where $0 < \alpha < 1$ indicates the relative position of the solar peak within a cycle.

(a) Make a rough sketch of the function for $\alpha = 1/3$. [5]

(b) Specify the constant term d of the Fourier series expansion of $f(t)$ and justify your answer based on the sketch. [5]

(c) What is the Fourier series expansion of $f(t)$ when $\alpha = \frac{1}{2}$? [5]

(d) For $\alpha \neq \frac{1}{2}$, it can be shown that the Fourier series coefficients decay as $O(n^{-3})$. Explain why this is the case. [8]

(e) The following two periodic functions are identical to $f(t)$ for $0 \le t \le T$

$$f_1(t) = A + \frac{A}{\pi}\sum_{n=1}^{\infty} u_n \sin(\pi n\alpha)\cos\left(\frac{\pi n t}{T}\right)$$

$$f_2(t) = \frac{A}{\pi}\sum_{n=1}^{\infty} (v_n + w_n\cos(\pi n\alpha))\sin\left(\frac{\pi n t}{T}\right)$$

for suitably defined u_n, v_n and w_n. Make a rough sketch of $f_1(t)$ and of $f_2(t)$ over one of their respective periods. [7]

12 (**long**) Consider the coupled system of differential equations

$$\begin{aligned} \frac{dy}{dt} + 4x &= u(t) \\ \frac{dx}{dt} - y &= w(t) \end{aligned}$$

(a) Show that the Laplace transform $\overline{x}(s)$ of $x(t)$ satisfies

$$\overline{x}(s) = \frac{\overline{u}(s) + s\overline{w}(s)}{s^2+4} + \frac{y(0) + sx(0)}{s^2+4}$$

where $\overline{u}(s)$ and $\overline{w}(s)$ are the Laplace transforms of $u(t)$ and $w(t)$, respectively. [6]

(b) By using Laplace transforms and not otherwise, determine $x(t)$ and $y(t)$ with initial conditions $x(0) = y(0) = 0$, when $u(t) = 0$ for all t and $w(t)$ is the Heaviside step function, i.e. $w(t) = H(t) = \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$. [8]

(c) By using Laplace transforms and not otherwise, compute

$$\int_0^t g_1(\tau) g_2(t-\tau)\, d\tau$$

for $g_1(t) = \begin{cases} \cos(2t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$ and $g_2(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$. [8]

(d) Determine $x(t)$ with initial conditions $x(0) = y(0) = 0$, when $u(t) = 0$ for all t and $w(t) = \begin{cases} -2t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$. [8]

END OF PAPER

Part IA 2025

Paper 4: Mathematical Methods

Sections B and C: Numerical Answers

9. (a) (i) $y = x - 2$, line with slope 1 and intercept -2.

   (ii) $x^2 + (y-4)^2 = 16$, a circle with radius 4 centred at $(0,\ 4)$

   (b) $\displaystyle\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right) + c$

   (c) (i) $z = \sqrt{2}e^{i\theta}$, $\theta \in \{\frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}\}$

   (ii) $z = \sqrt{2}e^{i\theta}$, $\theta \in \{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}\}$

10. (a) Eigenvalues $1-a$, a and $1+a$, eigenvectors $(-1,1,0)$, $(0,0,1)$ and $(1,1,0)$

    (b) (i) $\mathbf{A}^n = \dfrac{1}{2}\begin{bmatrix} \lambda_1^n + \lambda_3^n & -\lambda_1^n + \lambda_3^n & 0 \\ -\lambda_1^n + \lambda_3^n & \lambda_1^n + \lambda_3^n & 0 \\ 0 & 0 & 2\lambda_2^n \end{bmatrix}$

    (ii) $e^{\mathbf{A}} = \begin{bmatrix} e\cosh a & e\sinh a & 0 \\ e\sinh a & e\cosh a & 0 \\ 0 & 0 & e^a \end{bmatrix}$

    (iii) $e^{\mathbf{A}} \approx \begin{bmatrix} e & ea & 0 \\ ea & e & 0 \\ 0 & 0 & 1+a \end{bmatrix}$

11. (b) $d = A$

    (c) $f(t) = A - A\cos\left(\dfrac{\pi t}{\alpha T}\right)$

12. (b) $x(t) = \dfrac{1}{2}\sin(2t)\ ,\ \ y(t) = \cos(2t) - H(t)$

    (c) $\displaystyle\int_0^t g_1(\tau)g_2(t-\tau)\,d\tau = \frac{1}{4}\left(H(t) - \cos(2t)\right)$

    (d) $x(t) = \dfrac{1}{2}\left(\cos(2t) - H(t)\right)$