EGT0
ENGINEERING TRIPOS PART IA

Tuesday 14 June 2022 9 to 12.10

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Section A: multiple choice supplementary booklet
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Questions 1–8: see multiple choice supplementary booklet.

SECTION B

9 (**long**)

(a) Find, with justification, the shortest distance between the lines $\mathbf{r}$ and $\mathbf{s}$ given by

$$\mathbf{r} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + u \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{s} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + v \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$ [10]

(b) Consider the following equation

$$(\mathbf{c}.\mathbf{a})\mathbf{a} + (\mathbf{c}.\mathbf{b})\mathbf{b} = \mathbf{p}$$

where $\mathbf{a}$ and $\mathbf{b}$ are known vectors which are not parallel.

(i) What is the value of $\mathbf{p}.(\mathbf{a} \times \mathbf{b})$? [4]

(ii) Explain carefully why $\mathbf{c}$ can be written as

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$$ [4]

(iii) Given

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$$

find all solutions for $\mathbf{c}$. [12]

10 (**long**)

The symmetric matrix **A** is given by

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 4a \end{bmatrix}$$

where a is a constant.

(a) Determine the eigenvalues and eigenvectors of **A**. [10]

(b) By expressing **A** as the product of two orthogonal matrices and a diagonal matrix, determine an expression for $\mathbf{A}^n$ for any positive integer n. [5]

(c) An approximate expression for $\mathbf{A}^n$ is required for large n and positive a. By examining the dominant eigenvalues, determine the different ranges of a for which $\mathbf{A}^n$ can be approximated. Give the different approximations for $\mathbf{A}^n$ with justification. [15]

SECTION C

11 (**long**)

(a) The response $y(t)$ of a linear system with input $f(t)$ is given by the convolution integral

$$y(t) = \int_0^t g(t-\tau) f(\tau) d\tau$$

where $g(t)$ is the impulse response. Both $f(t)$ and $g(t)$ are zero for $t < 0$. Show that

$$\int_0^t g(t-\tau) f(\tau) d\tau = \int_0^t f(t-\tau) g(\tau) d\tau$$ [3]

(b) The response of the linear system is governed by the differential equation

$$\frac{d^2y}{dt^2} + 14\frac{dy}{dt} + 49y = f(t)$$

(i) Compute the step response. [10]

(ii) Show that the impulse response is given by

$$g(t) = te^{-7t}$$ [2]

(iii) Find the response to an input

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

Sketch the response and mark its salient features. [15]

12 (**long**)

(a) Consider the gradient function

$$\nabla f = \left(ye^x + x^2 + 1\right)\mathbf{i} + \left(\frac{1}{(y+1)(y+2)} + e^x\right)\mathbf{j}$$

(i) Find the function $f(x,\ y)$. [9]

(ii) Calculate $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$. [3]

(iii) Determine the gradient of the function $f(x, y)$ at the origin along the curve $\mathbf{c} = (e^{3u} - 1)\,\mathbf{i} + u\,\mathbf{j}$ in the direction of increasing u. [8]

(b) A surface is defined by the equation

$$\mathbf{r} = u^2 v\,\mathbf{i} + v^2\,\mathbf{j} + uv\,\mathbf{k}$$

Find the normal to the surface at the point where $u = 1$ and $v = 2$. [10]

END OF PAPER