Paper 4: Mathematical Methods

Solutions to 2024 Sections B and C

9. **Complex numbers**

(a) We rearrange

$$\begin{aligned} s(z+1) &= 2(z-1) \\ z(2-s) &= s+2 \\ z &= \frac{2+s}{2-s} = \frac{1+s/2}{1-s/2} \end{aligned}$$

[3]

(b) We substitute $s = ix$ into the inverse transform equation and compute the magnitude as

$$|z| = \left|\frac{1+ix/2}{1-ix/2}\right| = \frac{|1+ix/2|}{|1-ix/2|} = \frac{\sqrt{1+x^2/4}}{\sqrt{1+x^2/4}} = 1$$

Hence z is on the unit circle. [8]

(c)

$$\begin{aligned} s &= 2\frac{e^{i\theta}-1}{e^{i\theta}+1} = 2\frac{e^{i\theta/2}(e^{i\theta/2}-e^{-i\theta/2})}{e^{i\theta/2}(e^{i\theta/2}+e^{-i\theta/2})} \\ &= 2\frac{2i\sin(\theta/2)}{2\cos(\theta/2)} = 2i\tan(\theta/2) \end{aligned}$$

The result is on the imaginary axis. [8]

(d) We solve

$$\begin{aligned} 2\left(\frac{z-1}{z+1}\right) &= 2z \\ z-1 &= z(z+1) \\ z^2 &= -1 \end{aligned}$$

Expressing z as $z = \rho e^{i\theta}$, this becomes

$$\rho^2 e^{2i\theta} = e^{i\pi}$$

which gives the solution $\rho = 1$ and

$$\theta = \frac{\pi}{2} + n\pi$$

for any n. This gives two distinct doubling points: $e^{i\pi/2} = i$ and $e^{-i\pi/2} = -i$. [5]

(e) Using the binomial expansion, we write

$$\begin{aligned} \frac{1+s/2}{1-s/2} &= (1+s/2)\left(1+s/2+s^2/4+\mathcal{O}(s^3)\right) \\ &= 1+s+s^2/2+\mathcal{O}(s^3) \end{aligned}$$

which is identical to the power expansion of e^s up to and including the quadratic term. [6]

10. **Differential equations**

(a) For $f(t) = 0$:

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0$$

Auxiliary equation:

$$\frac{1}{q}\lambda^2 + \frac{p}{q}\lambda + 1 = 0 \Rightarrow \lambda = \frac{-\frac{p}{q} \pm \sqrt{(\frac{p}{q})^2 - 4\frac{1}{q}}}{2\frac{1}{q}}$$

(i) $\underline{p^2 - 4q > 0}$

Discriminant: $(\frac{p}{q})^2 - 4\frac{1}{q} > 0$

Two real roots: λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$

$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

(ii) $\underline{p^2 - 4q = 0}$

Discriminant: 0

Two identical roots: $-p/2$

$x(t) = (At + B)e^{-pt/2}$

(iii) $\underline{p^2 - 4q < 0}$

No real roots, two complex roots: $\lambda = \alpha \pm i\beta$

$x(t) = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$ [10]

(b) $\frac{d^2x}{dt^2} + 4x = 2\sin t + \sin 2t$

Homogeneous equation:

$\frac{d^2x}{dt^2} + 4x = 0$

Characteristic equation:

$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

General solution to homogeneous equation:

$x_h(t) = C_1\cos 2t + C_2\sin 2t$

Now to find a particular solution where $f(t) = 2\sin t + \sin 2t$. Note that in this case, $\sin 2t$ appears in both the right hand side and the complementary function. This can be dealt with by multiplying through by t in our trial solution.

So, let's guess $x_p(t) = \alpha\sin t + \beta\cos t + t(\gamma\sin 2t + \delta\cos 2t)$

Since the right hand side is odd, x_p must be odd such that the differential operator also produces an odd function.

Therefore, we need only try $x_p(t) = \alpha\sin t + t\delta\cos 2t$

$dx_p/dt = \alpha\cos t + \delta(\cos 2t - 2t\sin 2t)$

$d^2x_p/dt^2 = -\alpha\sin t - 4\delta\sin 2t - 4\delta t\cos 2t$

Substituting into the differential equation:

$-\alpha \sin t - 4\delta \sin 2t - 4\delta t \cos 2t + 4[\alpha \sin t + t\delta \cos 2t] = 2 \sin t + \sin 2t$

Grouping terms:

$[\sin(t)] \quad -\alpha + 4\alpha = 2 \Rightarrow \alpha = 2/3$

$[\sin(2t)] \quad -4\delta = 1 \Rightarrow \delta = -1/4$

Therefore, $x_p(t) = \frac{2}{3}\sin t - \frac{1}{4}t\cos 2t$

General solution is:

$x(t) = x_h(t) + x_p(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{2}{3}\sin t - \frac{1}{4}t\cos 2t$

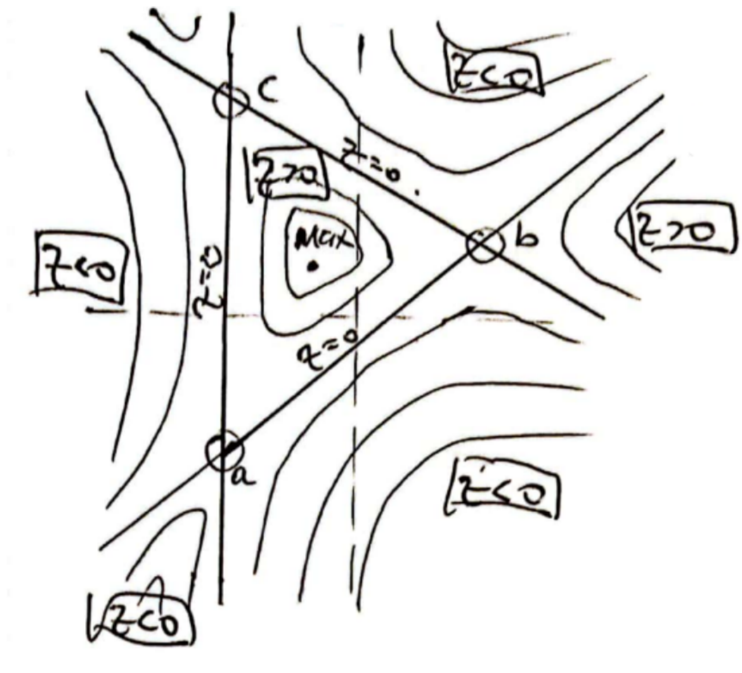
$x(0) = 0 \Rightarrow C_1 = 0$

$\left.\frac{dx}{dt}\right|_0 = 0 \Rightarrow 2C_2 + \frac{2}{3} - \frac{1}{4} = 0 \Rightarrow C_2 = -\frac{5}{24}$

$x(t) = \frac{2}{3}\sin t - \frac{5}{24}\sin 2t - \frac{1}{4}t\cos 2t$ [20]

11. **Functions of two variables, contour plots, gradients**

(a)



Contours for $z = 0$ are $x + 1 = 0$, $x + 2y - 2 = 0$ and $3x - 4y - 1 = 0$. [8]

(b) The partial derivative with respect to y is best found directly from the factorized expression, without multiplying out.

$$\begin{aligned}\frac{\partial z}{\partial y} &= (x+1)\left[(x+2y-2)(-4) + (3x-4y-1)(2)\right] \\ &= (x+1)\left[(-4x-8y+8) + (6x-8y-2)\right] = (x+1)(2x-16y+6) \\ &= 2(x+1)(x-8y+3)\end{aligned}$$

To find the partial derivative with respect to x, we first multiple out the factors

$$z = 3x^3 + 2x^2y - 4x^2 - 8xy^2 + 8xy - 5x - 8y^2 + 6y + 2$$

and then differentiate

$$\frac{\partial z}{\partial x} = 9x^2 + 8y + 4xy - 8y^2 - 8x - 5 \qquad [6]$$

(c) $\frac{\partial z}{\partial y} = 0$ implies either $x = -1$ or $x = 8y - 3$. We take each of these possibilities in turn and see when $\frac{\partial z}{\partial x} = 0$ as well.
$x = -1 \Rightarrow \frac{\partial z}{\partial x} = 9 + 8y - 4y - 8y^2 + 8 - 5 = -4(2y-3)(y+1) = 0$
$\Rightarrow y = -1, 3/2$
So there are stationary points at $(-1,-1)$ and $(-1,3/2)$.
$x = 8y - 3 \Rightarrow \frac{\partial z}{\partial x} = 9(64y^2 - 48y + 9) + 8y + 4(8y-3)y - 8y^2 - 8(8y-3) - 5 = 100(6y^2 - 5y + 1) = 100(3y-1)(2y-1) = 0 \Rightarrow y = 1/3, 1/2$
So there are stationary points at $(1, 1/2)$ and $(-1/3, 1/3)$. By inspection of the contour plot, the points $(-1,-1)$, $(-1,3/2)$ and $(1,1/2)$ lie at the intersections of the $z = 0$ contours and are therefore saddle points. The point $(-1/3, 1/3)$ lies in a region of positive z and is therefore a maximum. Note that the coordinates of the saddle points can be found directly from the contour plot, with no need for differentiation. [8]

(d) Evaluating the gradient at the origin:

$$\nabla z\Big|_{(0,0)} = (-5, 6)$$

This vector points in the direction of steepest ascent. Therefore, the contour of constant z passes through the origin at right angles to this, i.e. in the direction $(6,5)$ or $(-6,-5)$. [4]

(e) Now we evaluate the gradient at $(1,1)$:

$$\nabla z\Big|_{(1,1)} = (0, -16)$$

The rate of change of z in the direction **n** is given by $\nabla z \cdot \hat{\mathbf{n}} = (0,-16)\cdot(-1,-1)/\sqrt{2} = 16/\sqrt{2} = 8\sqrt{2}$. [4]

12. **Laplace transforms**

(a) The first differential equation is the differential equation that characterises a capacitor, i.e. the current is proportional to the derivative of the voltage. The second differential equation applies Kirchhoff's loop rule, i.e. the sum of voltages over the components R, L and C has to equal the source voltage V. For the inductance, the voltage is proportional to the derivative of the current, whereas the resistor satisfies Ohm's law where voltage is proportional to current. The source voltage is effectively zero for negative times because the switch is open, and only becomes V for times $t \geq 0$, hence the Heaviside function that multiplies V on the right of the equation. [2]

(b) We take the Laplace transforms of the equations to yield

$$\begin{cases} \bar{i}(s) = Cs\bar{v}(s) \\ R\bar{i}(s) + Ls\bar{i}(s) + \bar{v}(s) = \frac{V}{s} \end{cases}$$

The first equation gives

$$\bar{v}(s) = \frac{\bar{i}(s)}{Cs}$$

and substituting into the second equation, we obtain

$$\bar{i}(s) = \frac{\frac{V}{s}}{R + Ls + \frac{1}{Cs}} = \frac{CV}{1 + RCs + LCs^2} = \frac{V/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

[6]

(c) For $LC = 1$ and $RC = 2.5$, we have $R/L = \frac{RC}{LC} = 2.5$ and hence

$$\begin{aligned}\bar{i}(s) &= \frac{V/L}{s^2 + \frac{5}{2}s + 1} \\ &= \frac{V/L}{(s+2)(s+\frac{1}{2})} \\ &= \frac{2V}{3L}\left(\frac{1}{s+\frac{1}{2}} - \frac{1}{s+2}\right)\end{aligned}$$

Hence,

$$i(t) = \frac{2V}{3L}(e^{-t/2} - e^{-2t})$$

for $t \geq 0$ and $i(t) = 0$ for $t < 0$. [7]

(d) We have

$$\bar{v}(s) = \frac{\bar{i}(s)}{Cs} = \frac{V}{LC} \cdot \frac{1}{s\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

For $RC = LC = 4/5$, we obtain

$$\begin{aligned}\bar{v}(s) &= \frac{5V}{4} \cdot \frac{1}{s\left(s^2 + s + \frac{5}{4}\right)} \\ &= V \cdot \left(\frac{1}{s} - \frac{s+1}{s^2 + s + \frac{5}{4}}\right) \\ &= V \cdot \left(\frac{1}{s} - \frac{s+1}{\left(s+\frac{1}{2}\right)^2 + 1}\right) \\ &= V \cdot \left(\frac{1}{s} - \frac{s+1/2}{\left(s+\frac{1}{2}\right)^2 + 1} - \frac{1/2}{\left(s+\frac{1}{2}\right)^2 + 1}\right)\end{aligned}$$

Hence

$$v(t) = V\left[1 - e^{-t/2}\left(\cos t + \frac{1}{2}\sin t\right)\right]$$

for $t \geq 0$ and $v(t) = 0$ for $t < 0$. [8]

(e) For $i(t)$ to take the form $\beta t e^{-at}$, its Laplace transform must be of the form

$$\bar{i}(s) = \frac{\beta}{(s+a)^2}$$

For $LC = 1$, we know that

$$\bar{i}(s) = \frac{V/L}{s^2 + \frac{R}{L}s + 1}$$

This expression becomes

$$\bar{i}(s) = \frac{V/L}{(s+1)^2}$$

when $\frac{R}{L} = \frac{RC}{LC} = 2$. So we need $RC = 2$, for which the current is $i(t) = \frac{V}{L} t e^{-t}$ and hence $\beta = V/L$ and $a = 1$. [7]

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