Version JS/BBS/2

EGT0
ENGINEERING TRIPOS PART IA

Tuesday 11 June 2024 9 to 12.10

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
Section A: multiple choice supplementary booklet
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Questions 1–8: see multiple choice supplementary booklet.

SECTION B

9 (**long**) The bilinear transform is used in digital signal processing and maps the variable z to the variable s as

$$s = 2\left(\frac{z-1}{z+1}\right)$$

where both s and z are complex variables. For example, the point $z = 0$ is "transformed" to the point $s = -2$ in the complex plane.

(a) Show that the inverse transform expressing z in terms of s is $z = \frac{1+s/2}{1-s/2}$. [3]

(b) By expressing s as $s = ix$ for a real variable x, show that z is on the unit circle if s is on the imaginary axis. [8]

(c) Now expressing z as $z = e^{i\theta}$ for an angle θ, give an expression for the corresponding s and confirm that it is on the imaginary axis. [8]

(d) Find the doubling points of the bilinear transform, i.e., the points z for which $s = 2z$. [5]

(e) The inverse bilinear transform is an approximation for the exponential transform $z = e^s$. Show that this approximation is accurate up to the quadratic term. [6]

10 (**long**) Consider the following non-homogeneous second order differential equation

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = f(t)$$

where p and q are real coefficients and $f(t)$ is a real function of time t.

(a) For $f(t) = 0$, find the general solution for the following three cases:

(i) $p^2 - 4q > 0$

(ii) $p^2 - 4q = 0$

(iii) $p^2 - 4q < 0$ [10]

(b) Find the specific solution when $f(t) = 2\sin(t) + \sin(2t)$, $p = 0$, $q = 4$ and $x = dx/dt = 0$ at $t = 0$. [20]

SECTION C

11 (**long**) The height z of some terrain is given by the equation

$$z = (x + 1)(x + 2y - 2)(3x - 4y - 1)$$

(a) Sketch contours of constant z for the region $-2 < x < 2$ and $-2 < y < 2$. [8]

(b) Show that

$$\frac{\partial z}{\partial x} = 9x^2 + 8y + 4xy - 8y^2 - 8x - 5$$
$$\frac{\partial z}{\partial y} = 2(x + 1)(x - 8y + 3)$$ [6]

(c) Find the stationary points of z and classify them by inspection, without computing second derivatives. [8]

(d) In which direction does the contour of constant z pass through the origin? [4]

(e) Find the rate of change of z at the point $(x, y) = (1, 1)$ in the direction $(-1, -1)$. [4]

 (TURN OVER

12 (**long**) Consider the circuit in Fig. 1 comprising a voltage source V, a switch, a resistor R, an inductor L and a capacitor C. The switch is open at times $t < 0$ and closed for $t \geq 0$. The function $i(t)$ represents the current through the circuit and the function $v(t)$ represents the potential difference across the capacitor.

(a) Explain why $i(t)$ and $v(t)$ satisfy the differential equations

$$\begin{aligned} i &= C\frac{dv}{dt} \\ Ri + L\frac{di}{dt} + v &= VH(t) \end{aligned}$$

where $H(t)$ is the Heaviside step function, i.e. $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$. [2]

(b) Show that the Laplace transform $\bar{i}(s)$ of $i(t)$ satisfies

$$\bar{i}(s) = \frac{V/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

noting that $i(0) = 0$ A and $v(0) = 0$ V. [6]

(c) Derive an expression for $i(t)$ when $LC = 1$ and $RC = 2.5$. [7]

Remark: the unit for LC is seconds squared and the unit for RC is seconds, but we omit units in order not to risk confusing units with the variable s used in the Laplace transform.

(d) Derive an expression for $v(t)$ when $LC = 0.8$ and $RC = 0.8$. [8]

(e) Given $LC = 1$, for what value of RC does $i(t) = \beta t e^{-at}$ for some constants a and β? What are a and β? [7]

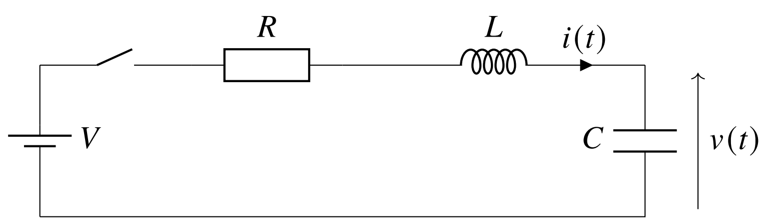

Fig. 1

END OF PAPER