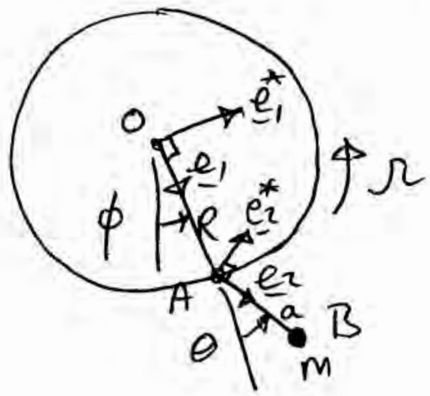
1/.

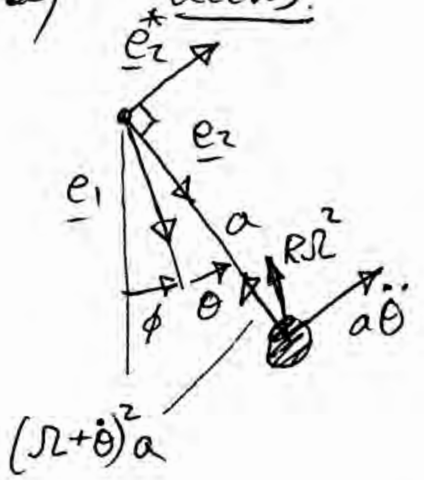


a) $\underline{OB} = R\underline{e}_1 + a\underline{e}_2$

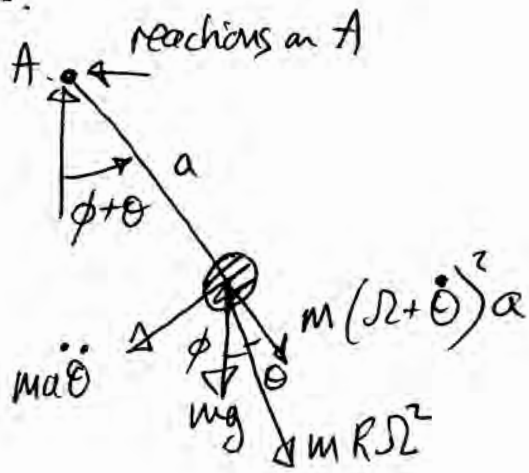
$\mathring{\underline{OB}} = R\Omega\underline{e}_1^* + a(\Omega+\dot{\theta})\underline{e}_2^*$

$\mathring{\mathring{\underline{OB}}} = -R\Omega^2\underline{e}_1 - a(\Omega+\dot{\theta})^2\underline{e}_2 + a(\cancelto{0}{\dot{\Omega}}+\ddot{\theta})\underline{e}_2^*$

b) accns.



forces.



moments about A ↻ $\quad \cancel{ma}\ddot{\theta}a + \cancel{mga}\sin(\theta+\phi) + \cancel{ma}R\Omega^2\sin\theta = 0.$

$$\therefore\ \underline{\ddot{\theta}a + g\sin(\theta+\phi) + R\Omega^2\sin\theta = 0}$$

c) i) when $\phi = 0$. $\quad \ddot{\theta}a + g\sin\theta + R\Omega^2\sin\theta = 0.$

small θ $\quad \ddot{\theta}a + (g+\Omega^2 R)\theta = 0.$

$$\therefore\ \underline{\underline{\omega_n\Big|_{\phi=0} = \sqrt{\frac{g+\Omega^2 R}{a}}}}$$

c) ii) when $\phi = \pi$, gondola hangs downwards because $\Omega < \sqrt{\frac{g}{R}}$
$\therefore$ equilibrium is $\theta = \pm\pi$. Let $\theta = \beta + \pi$ then find freq of small oscillations in β

$$\ddot{\beta}a + g\sin(\beta+\pi+\pi) + R\Omega^2\sin(\beta+\pi) = 0$$

$$\ddot{\beta}a + g\sin\beta - \Omega^2 R\sin\beta = 0.$$

$$\therefore\ \underline{\underline{\omega_n\Big|_{\phi=\pi} = \sqrt{\frac{g-\Omega^2 R}{a}}}}$$

c) iii) When $\phi = \frac{\pi}{2}$, equilibrium is not necessarily $\theta = -\frac{\pi}{2}$

Let equilibrium be $\theta = \theta_0$ when $\ddot{\theta} = 0$, $\phi = \frac{\pi}{2}$

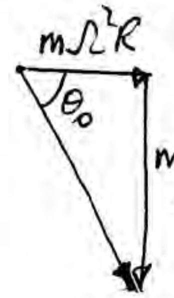
$$g\sin\left(\theta_0 + \frac{\pi}{2}\right) + R\Omega^2 \sin\theta_0 = 0$$

$$g\left(\sin\theta_0 \cancelto{0}{\cos\frac{\pi}{2}} + \cos\theta_0 \cancelto{1}{\sin\frac{\pi}{2}}\right) + R\Omega^2\sin\theta_0 = 0$$

$$g\cos\theta_0 = -R\Omega^2\sin\theta_0$$

$$\tan\theta_0 = -\frac{g}{\Omega^2 R}$$

$$\theta_0 = -\tan^{-1}\left(\frac{g}{\Omega^2 R}\right)$$



Let $\theta = \beta + \theta_0$, $\ddot{\theta} = \ddot{\beta}$, find freq of small oscillations w/ β

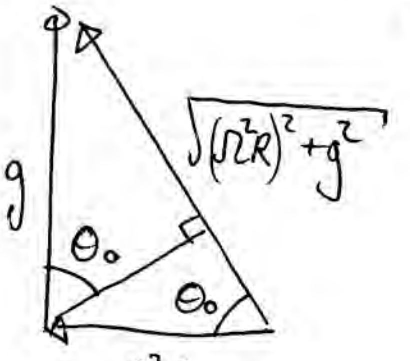
$$\ddot{\beta}a + g\sin\left(\beta + \theta_0 + \frac{\pi}{2}\right) + \Omega^2 R\sin(\beta + \theta_0) = 0$$

$$\ddot{\beta}a + g\left(\sin\beta\cos\left(\theta_0 + \frac{\pi}{2}\right) + \cos\beta\sin\left(\theta_0 + \frac{\pi}{2}\right)\right) + \Omega^2 R\left(\sin\beta\cos\theta_0 + \cos\beta\sin\theta_0\right) = 0$$

small β $\sin\beta \to \beta$, $\cos\beta \to 1$

$$\ddot{\beta}a + \beta\left(g\cos\left(\theta_0 + \frac{\pi}{2}\right) + \Omega^2 R\cos\theta_0\right) = -g\sin\left(\theta_0 + \frac{\pi}{2}\right) - R\Omega^2\sin\theta_0$$

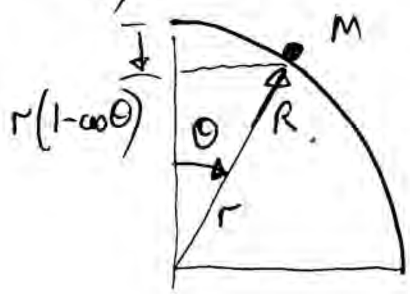
$$\ddot{\beta}a + \beta\underbrace{\left(-g\sin\theta_0 + \Omega^2 R\cos\theta_0\right)}_{=\sqrt{(\Omega^2 R)^2 + g^2}} = \underbrace{-g\cos\theta_0 - \Omega^2 R\sin\theta_0}_{=0}$$



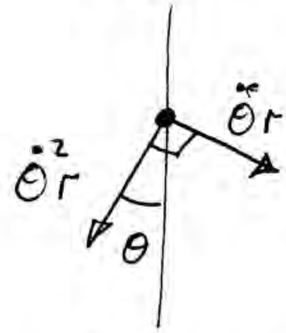
Hence
$$\boxed{\omega_n\Big|_{\phi = \frac{\pi}{2}} = \sqrt{\frac{\sqrt{(\Omega^2 R)^2 + g^2}}{a}}}$$

note that θ_0 has -ve value

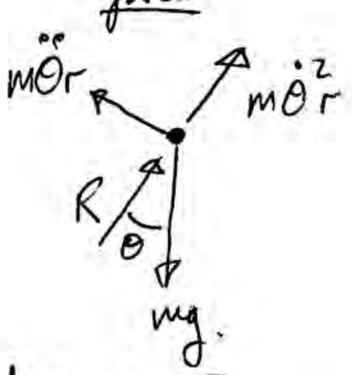
2/ a)



accns



forces



resolve forces perpendicular to surface at B

$$R + m\dot\theta^2 r - mg\cos\theta = 0.$$

loss of contact when $R=0$ $\therefore$ $m\dot\theta^2 r = mg\cos\theta$ — (1)

use energy to find $\dot\theta$ in terms of θ.

loss in PE = gain in KE.

$$\cancel{m}gr(1-\cos\theta) = \tfrac{1}{2}\cancel{m}(\dot\theta r)^2.$$

$$\dot\theta^2 r = 2g(1-\cos\theta)$$

hence from (1)

$$2\cancel{g}(1-\cos\theta) = \cancel{g}\cos\theta.$$

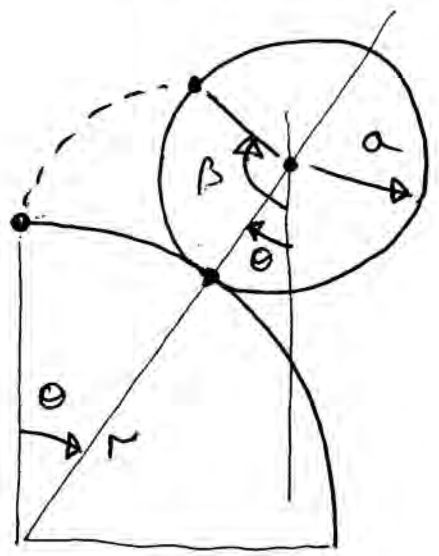
$$2 - 2\cos\theta = \cos\theta$$

$$\cos\theta = \frac{2}{3}.$$

$$\theta = \cos^{-1}\frac{2}{3} = 48.2^\circ$$

b) i) let angle of rotation of cylinder = β

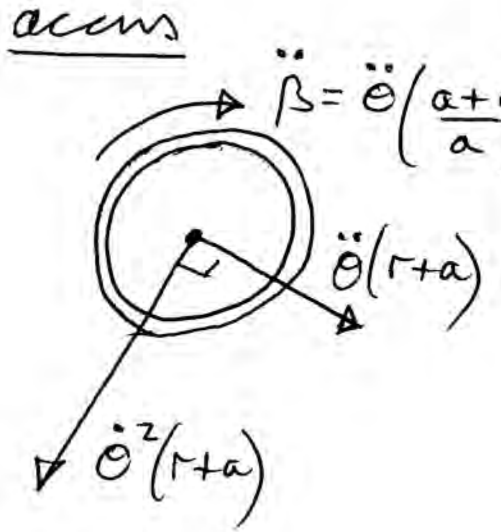
consider arc lengths to find relation between β and θ.



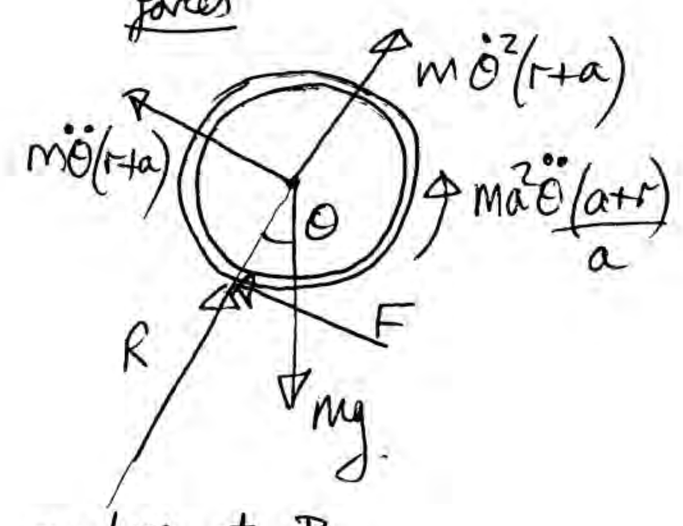
$$\beta a = a\theta + r\theta$$

$$\therefore \beta = \frac{\theta(a+r)}{a}.$$

accns



forces



resolve forces perpendicular to surface at B

$$R + m\dot{\theta}^2(r+a) - mg\cos\theta = 0.$$

$$R = mg\cos\theta - m\dot{\theta}^2(r+a) \quad (2)$$

gain of KE = loss of PE.

$$\frac{1}{2}\cancel{m}\dot{\theta}^2(r+a)^2 + \frac{1}{2}\underbrace{(ma^2)}_{\equiv I}\dot{\beta}^2 = \cancel{m}g(r+a)(1-\cos\theta)$$

$$\dot{\theta}^2(r+a)^{\cancel{2}} + \dot{\theta}^2\frac{(r+a)^{\cancel{2}}}{\cancel{a^2}}\cancel{a^2} = 2g\cancel{(r+a)}(1-\cos\theta)$$

$$\dot{\theta}^2 = \frac{g(1-\cos\theta)}{(r+a)}$$

put $\dot{\theta}^2$ into (2)

$$R = mg\cos\theta - mg\frac{(1-\cos\theta)}{\cancel{(r+a)}}\cancel{(r+a)}$$

set $R = 0$. $\quad 0 = \cos\theta - (1-\cos\theta)$

$$2\cos\theta = 1 \quad \therefore \quad \underline{\underline{\theta = \cos^{-1}\tfrac{1}{2} = 60^\circ}}$$

c) slip begins when $F = \mu R$, $\theta = \theta_s$

mts about centre of cylinder $F.a = ma^2\ddot{\theta}\frac{(a+r)}{a}$

$$F = m\ddot{\theta}(a+r)$$

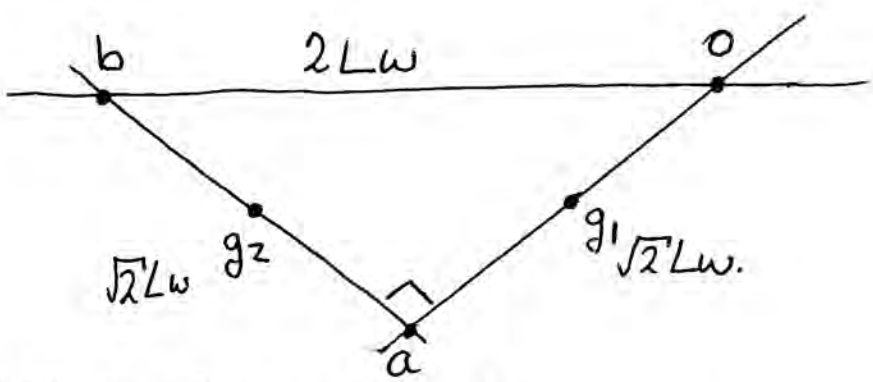
from (b) $\quad R = mg(2\cos\theta_s - 1)$

slip when $\quad m\ddot{\theta}(a+r) = \mu mg(2\cos\theta_s - 1)$

to get $\ddot{\theta}$ in terms of θ, resolve forces tangential to surface at B $\therefore$ $mg\sin\theta = F + m\ddot{\theta}(r+a) = 2m\ddot{\theta}(r+a)$

Hence $\frac{1}{2}\cancel{mg}\sin\theta_s = \mu\cancel{mg}(2\cos\theta_s - 1) \Rightarrow \underline{\underline{\mu = \frac{\sin\theta_s}{2(2\cos\theta_s - 1)}}}$

3/ a)

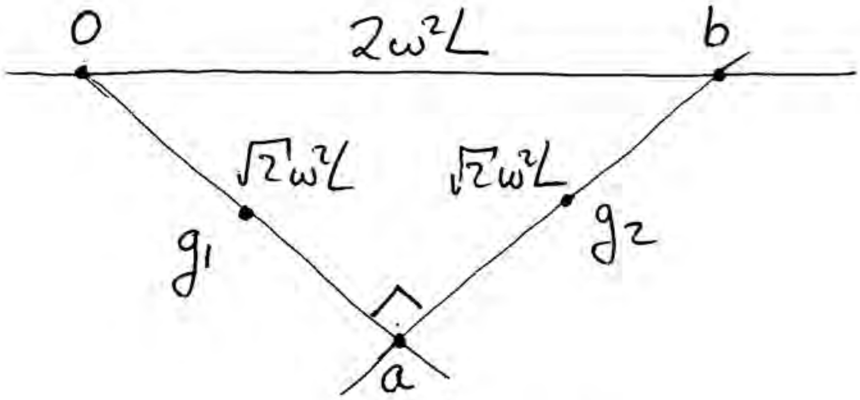


rels	↓	←
G_1	$L\omega/2$	$L\omega/2$
G_2	$L\omega/2$	$L\omega\,3/2$
B	0	$2L\omega$

$\omega_{AB} = \omega$

$\omega_{OA} = \omega$

b)

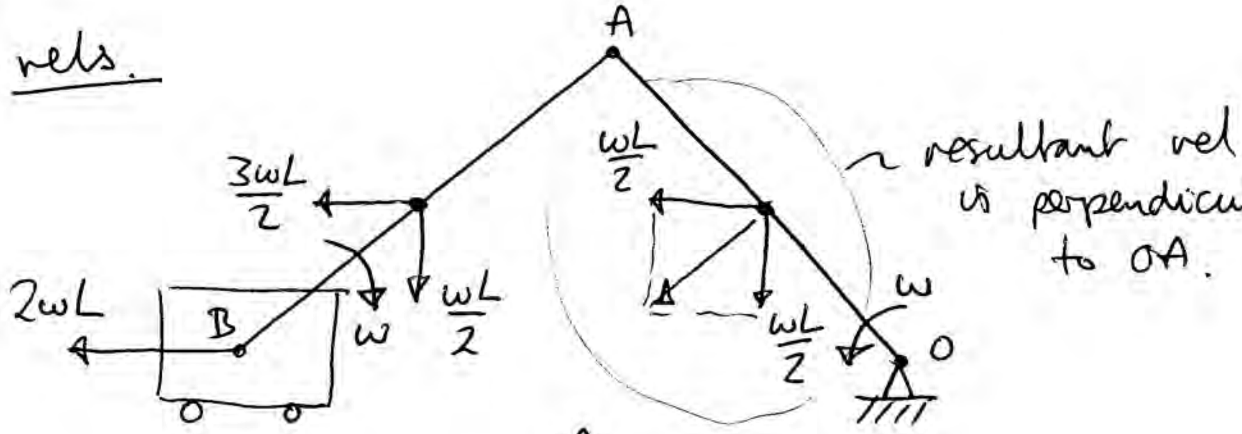


accns.	↓	→
G1	$\omega^2L/2$	$\omega^2L/2$
G2	$\omega^2L/2$	$\omega^2L\,\frac{3}{2}$
B	0	$2\omega^2L$.

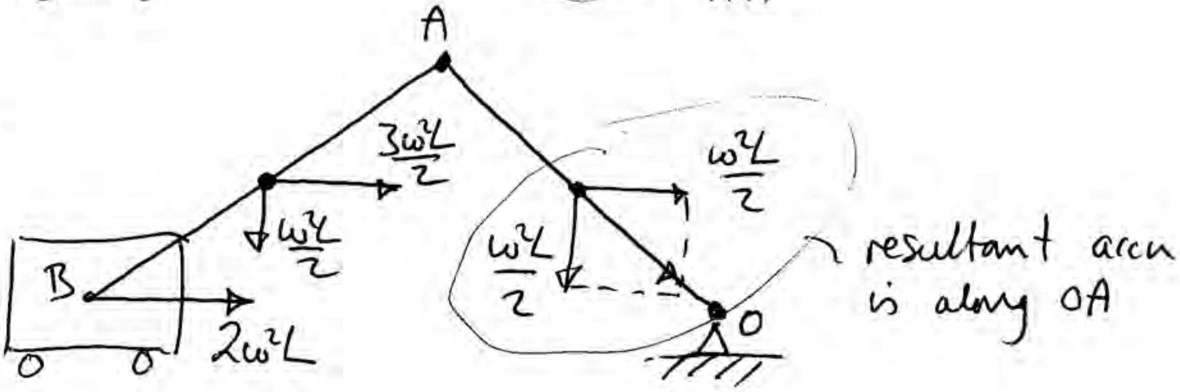
$\dot{\omega}_{AB} = 0$

$\dot{\omega}_{OA} = 0.$

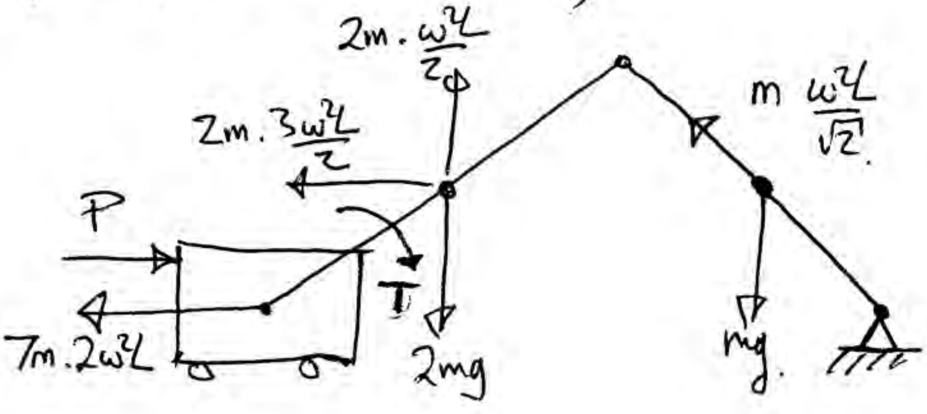
c) *vels.*



accns



forces (external and inertial)



virtual power.

$$T.\cancel{\omega} - P.2\cancel{\omega}L + 7m.2\omega^2L.2\cancel{\omega}L + 2mg.\frac{\cancel{\omega}L}{2} + 2m.\frac{3\omega^2L}{2}.\frac{3\cancel{\omega}L}{2}$$

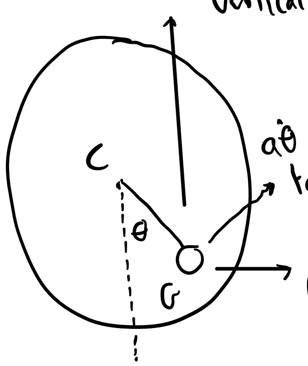
$$- 2m.\frac{\omega^2L}{2}.\frac{\cancel{\omega}L}{2} + mg.\frac{\cancel{\omega}L}{2} = 0$$

$$T = 2PL - \frac{3}{2}mgL - 32m\omega^2L^2$$

Part IB Paper 1 Section B : Cribs 2022

Vertical velocity $= a\dot{\theta}\sin\theta$

6. (a)



Horizontal vel $= a\dot{\theta}\cos\theta - R\dot{\theta}$

Due to the no slip condition, if the disc rotates by θ then it moves to the left a distance of $R\theta$

$$T = \tfrac{1}{2}M\left[(a\dot{\theta}\cos\theta - R\dot{\theta})^2 + (a\dot{\theta}\sin\theta)^2\right] + \tfrac{1}{2}J_G\dot{\theta}^2$$

⟵ Due to motion of CoG ⟶ Due to rotation about CoG

$$U = Mg(R - a\cos\theta) = Mg \times \text{ht of CoG above ground}$$

$\underline{L = T - U}$ as in question. [8]

(b) $$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0$$

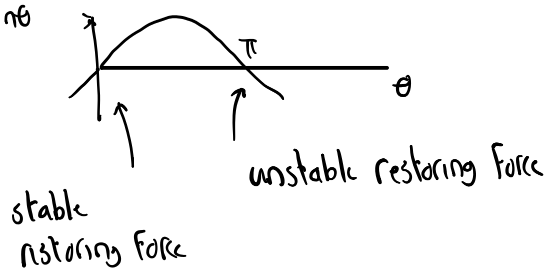
$$\frac{d}{dt}\left[M\dot{\theta}(a\cos\theta - R)^2 + M\dot{\theta}a^2\sin^2\theta + J_G\dot{\theta}\right]$$

$$-\left[-Ma\dot{\theta}\sin\theta\,(a\dot{\theta}\cos\theta - R\dot{\theta}) + Ma^2\dot{\theta}^2\sin\theta\cos\theta\right] + Mga\sin\theta = 0$$

$$M\ddot{\theta}(a\cos\theta - R)^2 - 2M\dot{\theta}^2 a\sin\theta\,(a\cos\theta - R) + M\ddot{\theta}a^2\sin^2\theta + M\dot{\theta}^2a^2\,2\sin\theta\cos\theta$$
$$+ J_G\ddot{\theta} + Ma\dot{\theta}^2\sin\theta\,(a\cos\theta - R) - Ma^2\dot{\theta}^2\sin\theta\cos\theta + Mga\sin\theta = 0$$

$$\ddot{\theta}\left[M(a^2+R^2-2aR\cos\theta)+J_G\right]+M\dot{\theta}^2 aR\sin\theta+Mga\sin\theta=0$$ [6]

(c) For equilibrium $\ddot{\theta}=\dot{\theta}=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$ or π



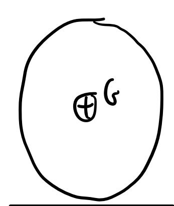
$\Rightarrow$ Stable equilibrium at $\theta=0$. For small oscillations:

$$\ddot{\theta}\left[M(a^2+R^2-2aR)+J_G\right]+Mga\theta=0$$

$$\Rightarrow \ddot{\theta}\left[M(a-R)^2+J_G\right]+Mga\theta=0$$

$$\Rightarrow \underline{\omega_n^2=\frac{Mga}{\left[M(a-R)^2+J_G\right]}}\quad =0 \text{ when } a=0$$

For $a=0$



G has a constant height $\Rightarrow$ no change in potential $\Rightarrow$ no restoring force.
The disc will simply roll on the surface with $\dot{\theta}=$ constant. [6]

(d) Require initial kinetic energy to exceed increase in potential energy from $\theta=0$ to $\theta=\pi$.

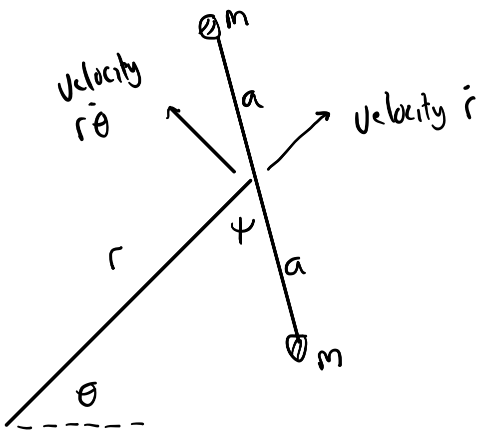
Initial $T=\dot{\theta}^2\left[\frac{1}{2}M(a-R)^2+\frac{1}{2}J_G\right]$

Initial $V=Mg(R-a)$ Final $V=Mg(R+a)$

$\Rightarrow$ Require $\quad \dot{\theta}^2 \left[\tfrac{1}{2} M (a-R)^2 + \tfrac{1}{2} J_G \right] > 2Mga$

$$\Rightarrow \quad \underline{\dot{\theta}^2 > 4Mga / \left[M(a-R)^2 + J_G \right]} \qquad [5]$$

5. (a)



$$T = \frac{1}{2} M_T v^2 + \frac{1}{2} J_G \dot{\phi}^2$$

rotation rate

total mass

Moment of inertia about centre of mass

velocity of centre of mass

Here: $M_T = 2m$

$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$\dot{\phi} = \dot{\theta} + \dot{\psi}$ (total rotational rate) and $J_G = 2ma^2$

$$\Rightarrow \underline{T = m(\dot{r}^2 + r^2\dot{\theta}^2) + ma^2(\dot{\theta} + \dot{\psi})^2}$$

This can also be obtained (longer route) by finding the velocity of each mass.

[8]

(b) For r equation:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{r}}\right] - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = 0$$

$$\frac{d}{dt}\left[2m\dot{r}\right] - 2mr\dot{\theta}^2 + \frac{2GMm}{r^2}$$ <u>neglecting</u> a^2 component as small

$$\Rightarrow \quad \underline{2m(\ddot{r} - r\dot{\theta}^2) = -\frac{2GMm}{r^2}}$$

$r = R =$ constant, then $\dot{\theta}^2 = GM/R^3 =$ constant also

Period $\underline{T_p = 2\pi/\dot{\theta} = 2\pi\sqrt{R^3/GM}}$ [8]

(c) For $\dot{\theta} =$ constant $= \Omega$ say and $r = R =$ constant then

$$T = m(R^2\Omega^2) + ma^2(\Omega + \dot{\psi})^2$$

$$V = -\frac{GMm}{R^3}\left[2R^2 + 3a^2\cos^2\psi\right]$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\psi}}\right] - \frac{\partial T}{\partial \psi} + \frac{\partial V}{\partial \psi} = 0$$

$$\Rightarrow \frac{d}{dt}\left[2ma^2(\Omega + \dot{\psi})\right] + \frac{GMm}{R^3}\left[6a^2\cos\psi\sin\psi\right] = 0$$

$$\Rightarrow \quad \underline{2ma^2\ddot{\psi} + \frac{6GMm}{R^3}a^2\cos\psi\sin\psi = 0}$$

For equilibrium $\sin\psi\cos\psi = 0 \Rightarrow \psi = 0$ or $\psi = \pi/2$

$$\frac{d}{d\psi}(\sin\psi\cos\psi) = \cos^2\psi - \sin^2\psi \;\; > 0 \text{ when } \psi = 0$$

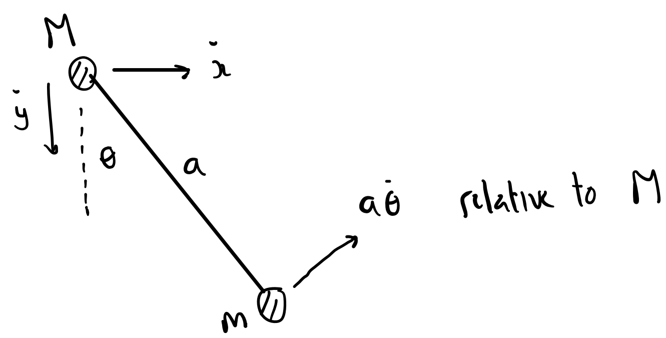
$$< 0 \text{ when } \psi = \pi/2$$

$\Rightarrow$ $\psi = 0$ is the stable equilibrium position.

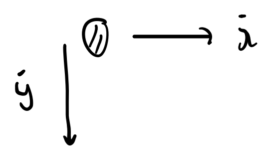
Linearise $\Rightarrow$ $2ma^2\ddot{\psi} + \frac{6GMm}{R^3}a^2\psi = 0 \Rightarrow \underline{\omega_n^2 = \frac{3GM}{R^3}}$

$\Rightarrow$ $\underline{T_n/T_p = 1/\sqrt{3}}$ [9]

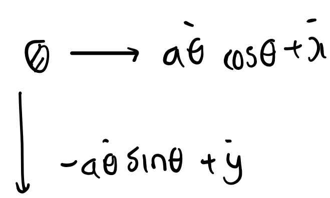
6. (a)



M :



m :



$$T = \tfrac{1}{2}M(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2}m\left[(a\dot{\theta}\cos\theta+\dot{x})^2 + (\dot{y}-a\dot{\theta}\sin\theta)^2\right]$$

$$\underline{T = \tfrac{1}{2}(M+m)(\dot{x}^2+\dot{y}^2) + ma\dot{\theta}\left[\dot{x}\cos\theta - \dot{y}\sin\theta\right] + \tfrac{1}{2}ma^2\dot{\theta}^2}$$

$$\underline{U = \tfrac{1}{2}k(x^2+y^2) - mg(a\cos\theta + y) - Mgy}$$ [6]

(b) For x $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} = 0$

$$\Rightarrow \frac{d}{dt}\left[(M+m)\dot{x} + ma\dot{\theta}\cos\theta\right] + kx = 0$$

$$\Rightarrow \underline{(M+m)\ddot{x} + ma\ddot{\theta}\cos\theta - ma\dot{\theta}^2\sin\theta + kx = 0}$$

For y $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial V}{\partial y} = 0$

$$\frac{d}{dt}\left[(M+m)\dot{y} + m(\dot{y} - a\dot{\theta}\sin\theta)\right] + ky - Mg = 0$$

$$\Rightarrow \quad \underline{(M+m)\ddot{y} - ma\ddot{\theta}\sin\theta - ma\dot{\theta}^2\cos\theta + ky = (M+m)g}$$

For θ $\quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$

$$\frac{d}{dt}\left[ma(\dot{x}\cos\theta - \dot{y}\sin\theta) + ma^2\dot{\theta}\right] + ma\dot{\theta}\dot{x}\sin\theta + ma\dot{\theta}\dot{y}\cos\theta + mga\sin\theta = 0$$

$$\Rightarrow \quad ma(\ddot{x}\cos\theta - \ddot{y}\sin\theta) - ma\dot{x}\dot{\theta}\sin\theta - ma\dot{y}\dot{\theta}\cos\theta + ma^2\ddot{\theta}$$
$$+ ma\dot{\theta}\dot{x}\sin\theta + ma\dot{\theta}\dot{y}\cos\theta + mga\sin\theta = 0$$

$$\Rightarrow \quad \underline{ma^2\ddot{\theta} + ma(\ddot{x}\cos\theta - \ddot{y}\sin\theta) + mga\sin\theta = 0}$$ [8]

(c) For $y=0$ and small θ:

$$(M+m)\ddot{x} + ma\ddot{\theta} + kx = 0$$

$$ma^2\ddot{\theta} + ma\ddot{x} + mga\theta = 0$$

$$\Rightarrow \quad \underline{\begin{pmatrix} M+m & ma \\ ma & ma^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & mga \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$ [$\varnothing$]

(d) $\quad |-\omega^2 M + k| = 0$

$$\Rightarrow \quad [-\omega^2 (M+m) + k][-\omega^2 ma^2 + mga] - \omega^4 m^2 a^2 = 0$$

$$\Rightarrow \omega^4 \overbrace{[\underbrace{ma^2 (M+m) - m^2 a^2}_{0 \text{ for } M=0}]}^{A} - \omega^2 \overbrace{[\underbrace{(M+m) mga + kma^2}_{m^2 ga + kma^2 \text{ for } M=0}]}^{B} + \overbrace{mgak}^{C} = 0$$

$$\omega^2 = \frac{1}{2A}\left\{ B \pm \sqrt{B^2 - 4AC} \right\} \quad \text{for } A \to 0 \quad \omega^2 = \infty \quad \text{or} \quad \omega^2 = C/B$$

Other solution $\quad \omega^2 = mgak / [m^2 ga + kma^2]$

$$\underline{\omega^2 = \frac{gk}{mg + ka}}$$

For $\omega^2 = \infty \qquad -\omega^2 \begin{pmatrix} M & ma \\ ma & ma^2 \end{pmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} -a \\ 1 \end{pmatrix}$

Mass does not move in this mode shape

$\Rightarrow$ zero kinetic energy, infinite frequency

[7]