EGT1
ENGINEERING TRIPOS PART IB

Monday 6 June 2022   09.00 to 11.10

Paper 1

MECHANICS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section. All questions carry the same number of marks.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

*Answer not more than **two** questions from this section*

1 A fairground ride consists of a large wheel mounted in a vertical plane with many gondolas, in which the passengers sit, suspended from the rim of the wheel. Figure 1 shows the wheel with radius R rotating about a fixed centre O at constant speed Ω in an anticlockwise direction, such that $\Omega < \sqrt{g/R}$, where g is the acceleration to due to gravity. Figure 1 shows only one of the many gondolas, connected to the rim of the wheel at point A. The gondola is a light rod AB of length a with a point mass m attached at B. End A is freely pivoted to the wheel. The instantaneous angle of OA to the vertical is ϕ and the instantaneous angle of AB to a line extended from OA is θ. All motion occurs in a fixed vertical plane.

(a) By defining two pairs of orthogonal unit vectors, derive vector expressions for the position, velocity and acceleration of B relative to O. [5]

(b) Use Newton's second law to derive the following equation of free vibration of the gondola:

$$\ddot{\theta}a + g\sin(\theta + \phi) + \Omega^2 R\sin\theta = 0.$$

[8]

(c) From the equation in (b) derive an expression for the natural frequency of small oscillations of the gondola when:

(i) $\phi = 0$ rad; [2]

(ii) $\phi = \pi$ rad; [4]

(iii) $\phi = \pi/2$ rad. [6]

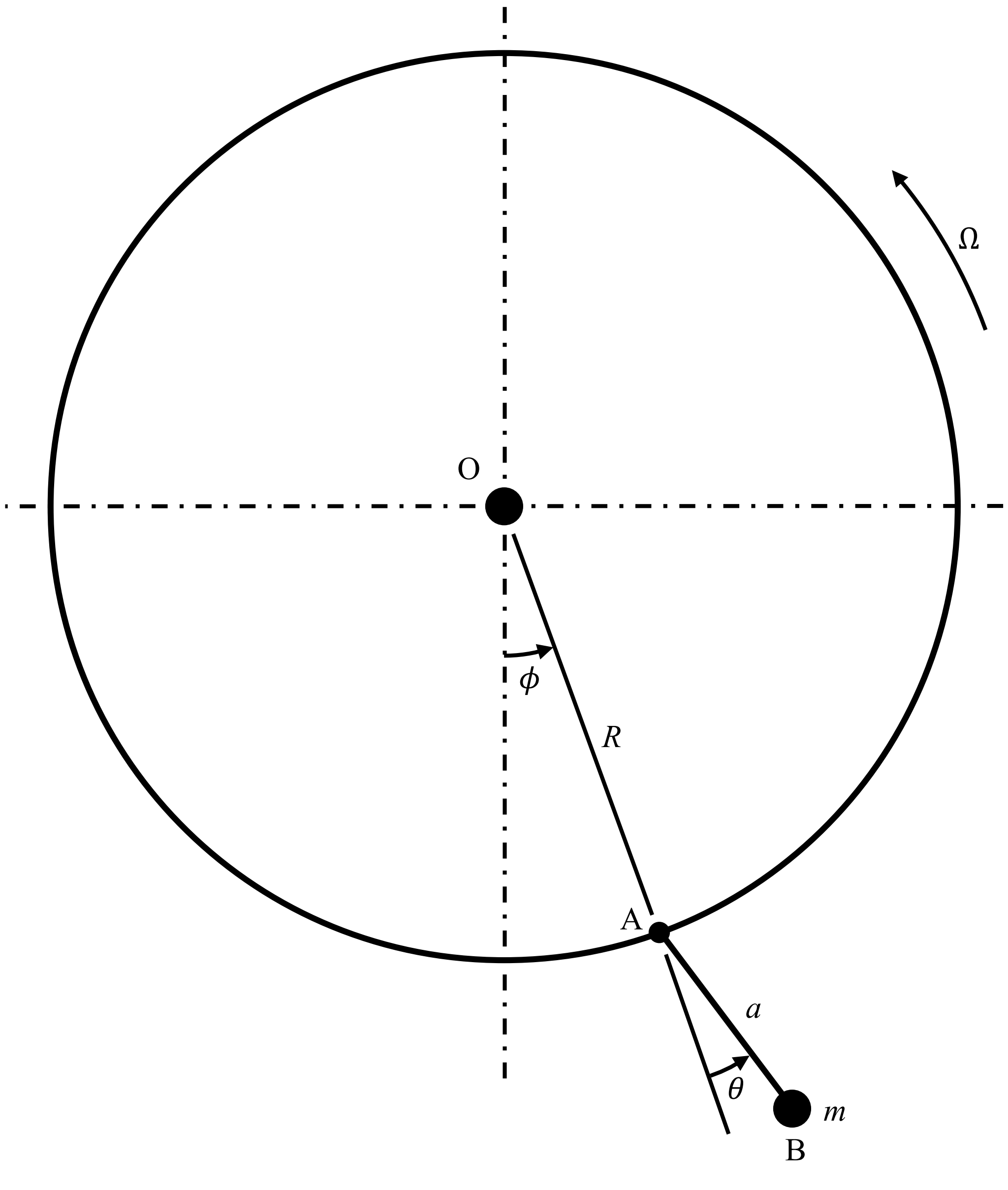

Fig. 1

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2 (a) Figure 2(a) shows a particle of mass m on a smooth frictionless surface of radius r, with centre of curvature at O. The particle is initially stationary at point A vertically above O. The particle then travels along the surface under the action of gravity, making contact with the surface at B, so that at an instant in time the angle AOB is θ. Find the angle θ at which the particle loses contact with the surface. All motion occurs in a fixed vertical plane. [6]

(b) Figure 2(b) shows a thin-walled cylinder of radius a and mass m on a rough surface of radius r, with centre of curvature at O. The cylinder is initially stationary at point A vertically above O. The cylinder then rolls along the surface under the action of gravity, making contact with the surface at B without slipping, so that at an instant in time the angle AOB is θ. All motion occurs in a fixed vertical plane.

(i) Find the angle θ at which the cylinder loses contact with the surface. [13]

(ii) If slip is observed to begin when $\theta = \theta_s$ find the corresponding coefficient of friction between the surface and the cylinder. [6]

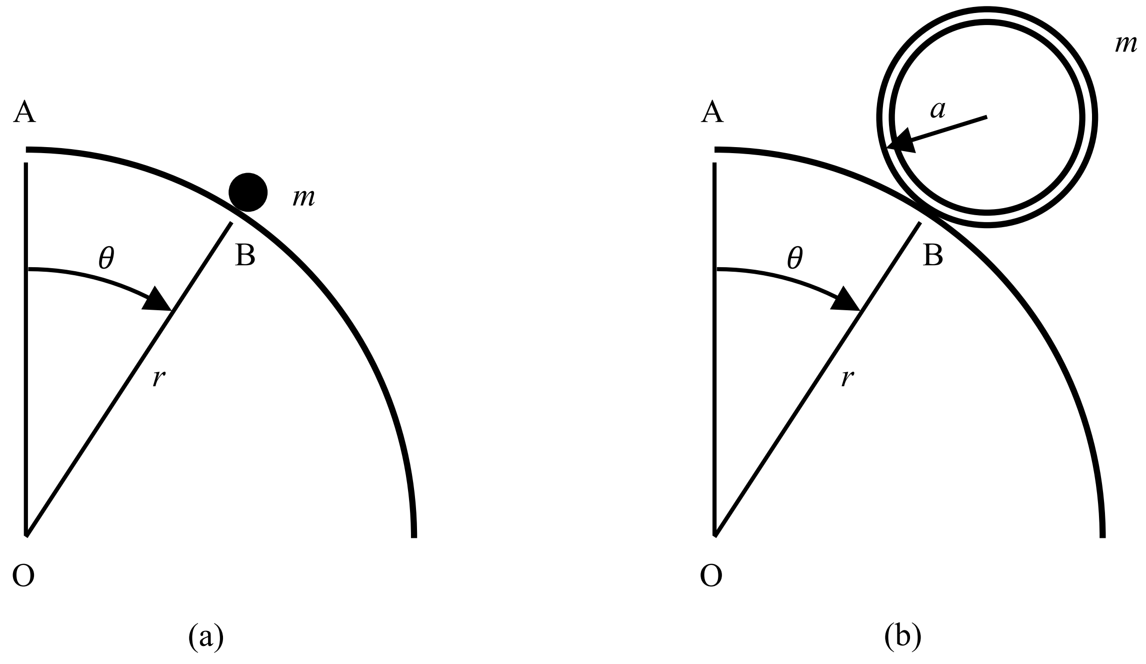

Fig. 2

3 Figure 3 shows an idealisation of a rower in side elevation. The block at B with mass $7m$ slides horizontally without friction and represents the upper body of the rower. The upper legs are represented by the uniform beam AB of length $\sqrt{2}L$ and mass $2m$ centred at G_2. The lower legs are represented by the uniform beam OA of length $\sqrt{2}L$ and mass m centred at G_1. Point O is ground. There are frictionless pivots at O, A and B, representing the ankles, knees and hips. At the instant shown angle OAB is $\pi/2$ rad. A force P acts horizontally on the rower's upper body. A muscle torque T is applied to the upper legs AB across the hip joint B. The angular velocity ω of the upper legs AB is constant. All motion occurs in a fixed vertical plane.

(a) Find the velocity of the body at B and the velocities at G_1 and G_2. [5]

(b) Find the acceleration of the body at B and the accelerations at G_1 and G_2. Confirm that the magnitude of the acceleration of the body at B is $2\omega^2 L$. [8]

(c) Find the torque T. [12]

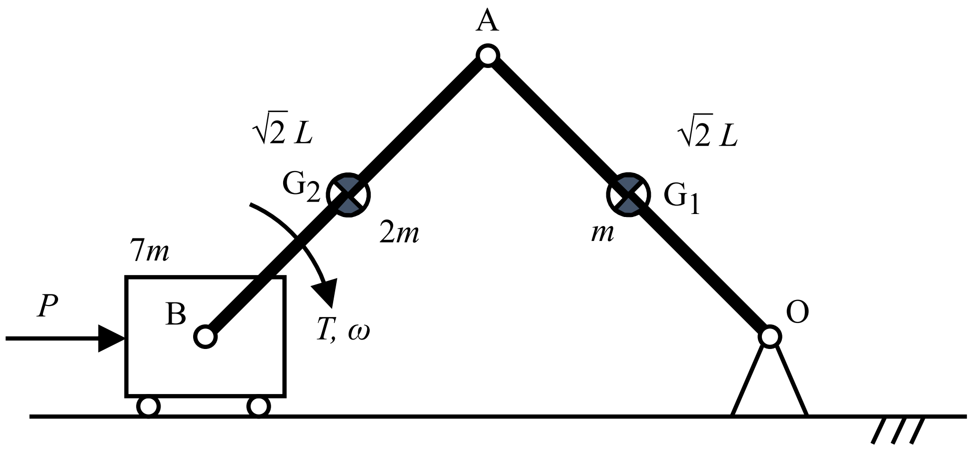

Fig. 3

SECTION B

*Answer not more than **two** questions from this section*

4 A rigid circular disc of mass M and radius R rolls without slipping over a horizontal plane, as shown in Fig. 4. The centre of gravity G of the disc is a distance a from the geometric centre C of the disc, and the moment of inertia of the disc about G is I_G. The motion of the disc is described by the angle $\theta(t)$ between the line CG and the vertical direction. All motion occurs in a fixed vertical plane.

(a) Show that the Lagrangian of the system can be written as

$$L = \frac{M}{2}\left[(R\dot{\theta} - a\dot{\theta}\cos\theta)^2 + (a\dot{\theta}\sin\theta)^2\right] + \frac{I_G}{2}\dot{\theta}^2 - Mg(R - a\cos\theta)$$

where g is the acceleration due to gravity. [8]

(b) Find the equation of motion of the system. [6]

(c) Identify the two equilibrium positions, and show that one position is stable and the other is unstable. Find the natural frequency for small oscillations around the stable equilibrium position. Explain in physical terms why the natural frequency is zero when $a = 0$. [6]

(d) The disk is initially in the position $\theta = 0$ and it is given an initial angular velocity $\dot{\theta}$. Find the minimum value of $\dot{\theta}$ required to ensure that the disk will roll along the plane without reversing direction. [5]

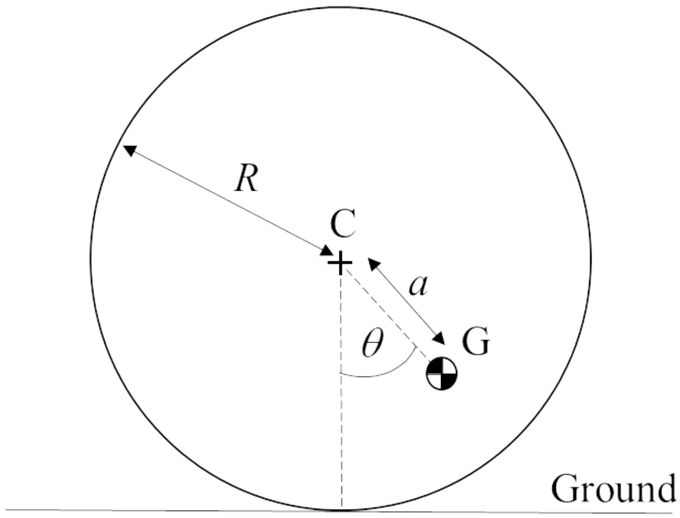

Fig. 4

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5 A satellite in orbit around the earth consists of two masses, each of mass m, that are separated by a light rigid rod of length $2a$, as shown in Fig. 5. The centre of gravity, A, of the satellite has polar coordinates $r(t)$ and $\theta(t)$ relative to the centre of the earth, C, and the rigid rod makes an angle $\psi(t)$ to the line CA. All motion occurs in a fixed plane.

(a) Neglecting the mass of the rigid rod, show that the kinetic energy of the satellite is given by

$$T = m(\dot{r}^2 + r^2\dot{\theta}^2) + ma^2(\dot{\theta} + \dot{\psi})^2.$$

[8]

(b) For the case where $a \ll r$ the potential energy of the satellite is given by

$$V = -\frac{GMm}{r^3}(2r^2 + 3a^2\cos^2\psi)$$

where G is the gravitational constant and M is the mass of the earth (note: you do *not* need to prove this result). Show that for $a \ll r$ the equation of motion that governs $r(t)$ is approximately

$$2m(\ddot{r} - r\dot{\theta}^2) = -\frac{2GMm}{r^2}.$$

Show that a circular orbit with constant radius $r = R$ is a solution to this equation, and find an expression for the orbital period in terms of R. [8]

(c) Find the equation of motion that governs the angle $\psi(t)$ for the case in which the centre of gravity has a circular orbit of radius R and constant $\dot{\theta}$. Hence find an expression for the natural frequency of small oscillations in $\psi(t)$ around the stable equilibrium position. Find the ratio between the period of the oscillations and the orbital period. [9]

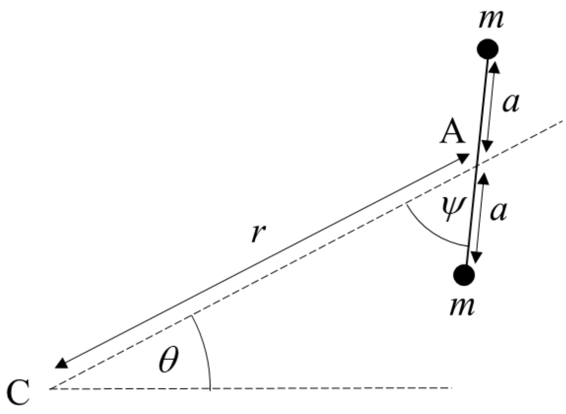

Fig. 5

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6 The system shown in Fig. 6 consists of a massless rigid rod of length a, which has a mass m attached at one end and a mass M attached at the other. The system moves in a fixed vertical plane, and the horizontal and vertical displacements of the mass M relative to a fixed point A are respectively $x(t)$ and $y(t)$. The angle of the rod to the vertical is $\theta(t)$. A spring of stiffness k couples the mass M to the fixed point A, and the unstretched length of the spring is zero (i.e. the spring has zero force when $x = y = 0$).

(a) Show that the kinetic energy of the system is

$$T = \frac{1}{2}(M+m)(\dot{x}^2+\dot{y}^2) + ma\dot{\theta}(\dot{x}\cos\theta - \dot{y}\sin\theta) + \frac{1}{2}ma^2\dot{\theta}^2$$

and find an expression for the potential energy, including the effect of gravity. [6]

(b) Derive the three equations of motion of the system. [8]

(c) If the system is restrained so that $y = 0$, linearise the two remaining equations of motion about the stable equilibrium position and show that they have the form

$$\begin{pmatrix} M+m & ma \\ ma & ma^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & mga \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

[4]

(d) Find an equation for the two natural frequencies of the system. Demonstrate that in the limit $M \to 0$ one of these frequencies becomes infinite, and find the other, finite, frequency. Find the mode shape associated with the infinite natural frequency, and explain your result in physical terms. [7]

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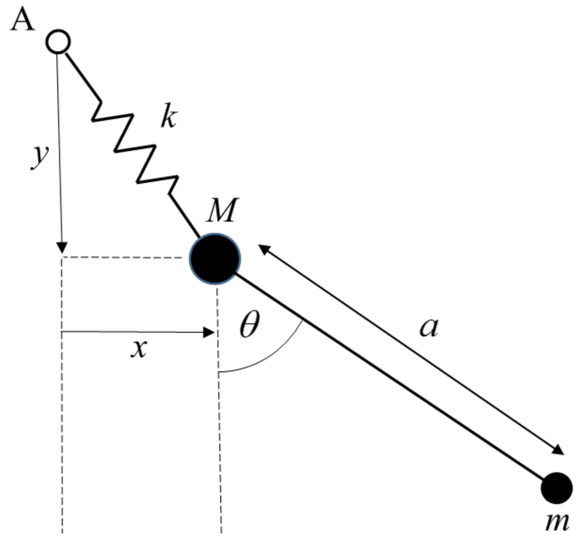

Fig. 6

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