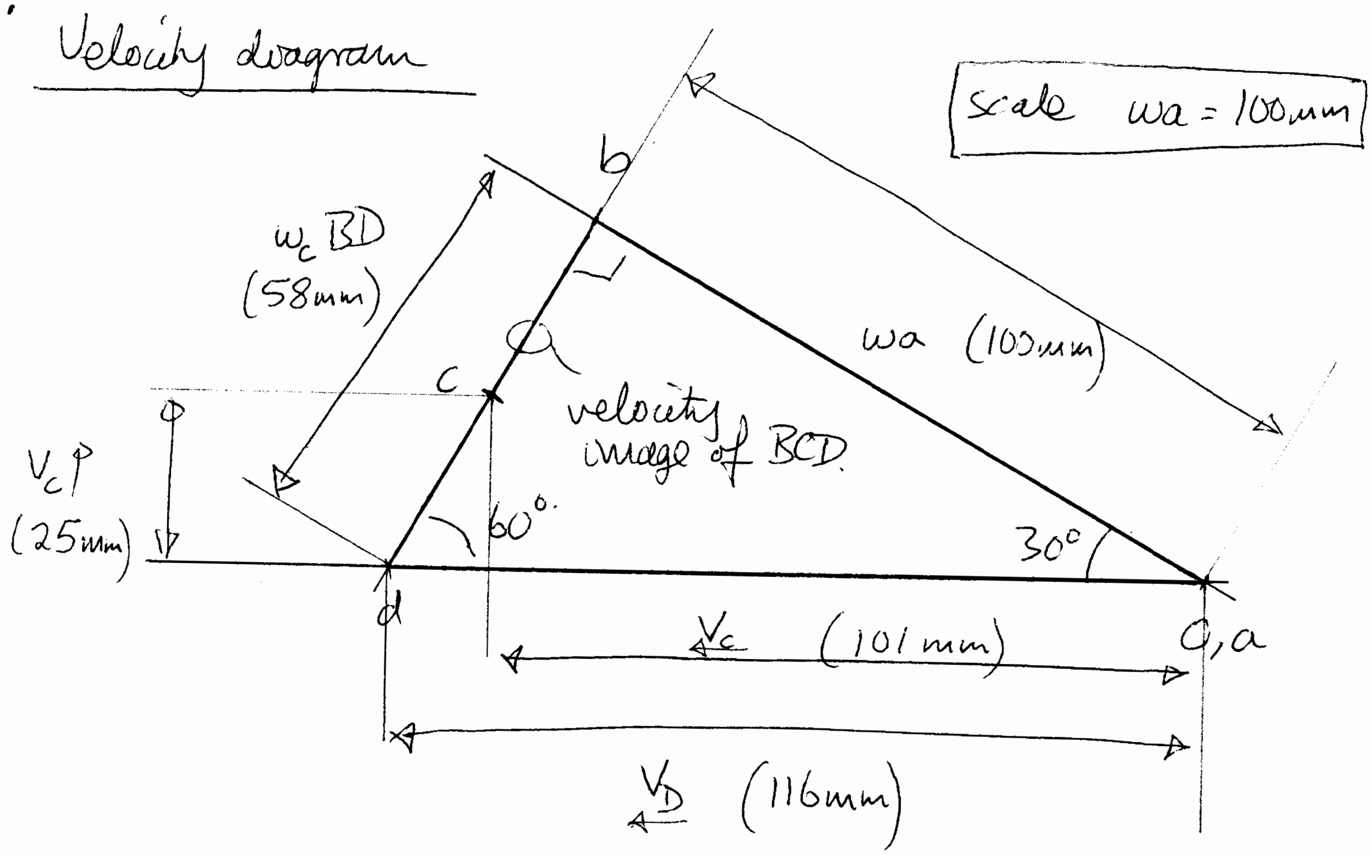
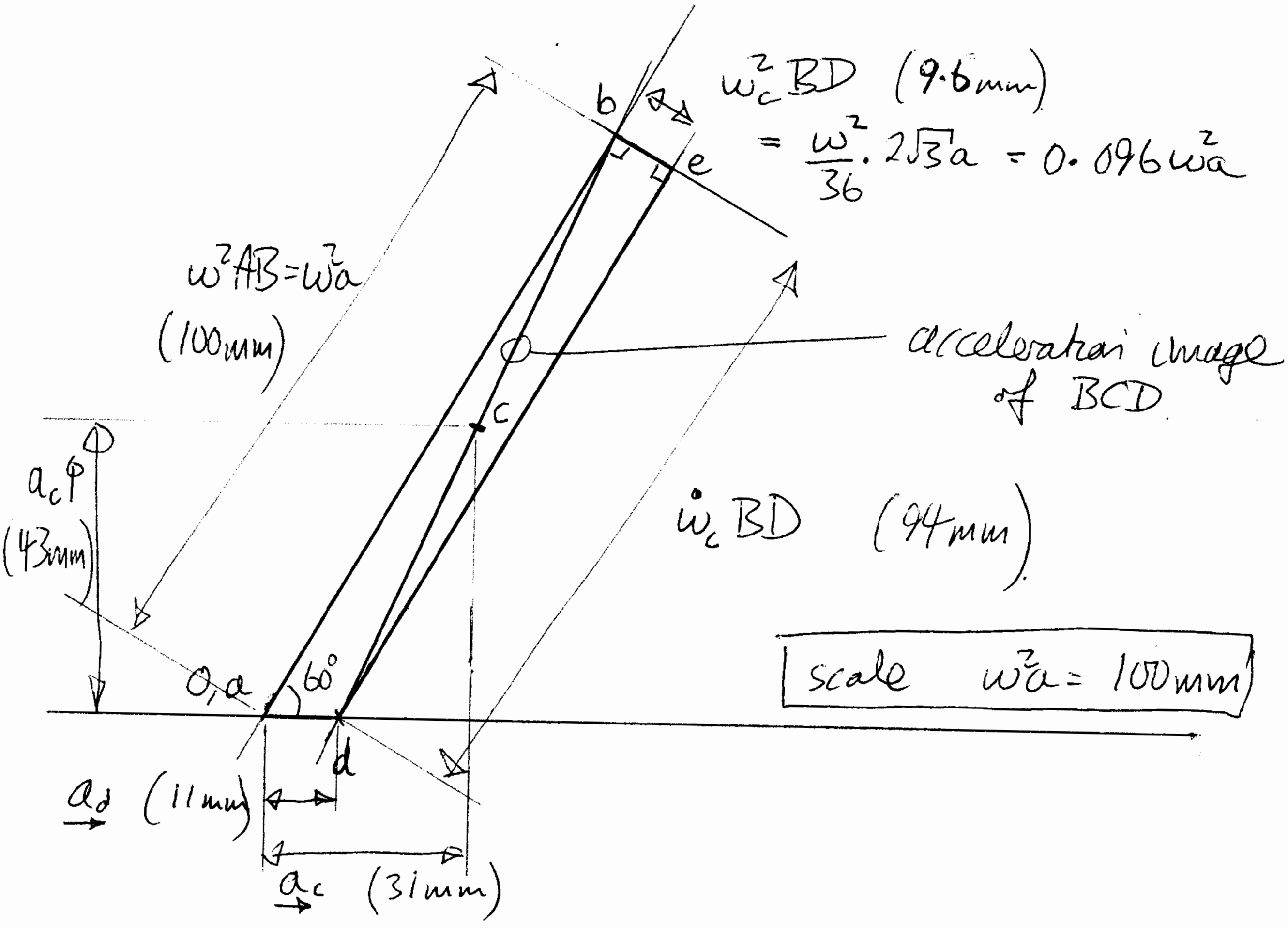
1. Velocity diagram



Acceleration diagram



1 a) see velocity diagram, measure directly:

$$\underline{V}_D = -\omega a . \frac{116mm}{100mm}\,\underline{i} = \underline{\underline{-1.16\,\omega a\,\underline{i}}}$$

$$\omega_c BD = \omega a . \frac{58mm}{100mm} .$$

$$\therefore\ \omega_c = \frac{0.58\,\omega a}{2\sqrt{3}\,a} = \underline{\underline{0.17\,\omega \circlearrowleft}} \quad (2.s.f.)$$

$$\underline{V}_c = -\omega a . \frac{101mm}{100mm} . \underline{i} + \omega a . \frac{25mm}{100mm}\,\underline{j} = \underline{\underline{-1.01\,\omega a\,\underline{i} + 0.25\,\omega a\,\underline{j}}}$$

[6]

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alternatively, use trigonometry:

$$\underline{V}_D = -\frac{\omega a\,\underline{i}}{\cos 30^\circ} = \underline{\underline{-\frac{2\omega a}{\sqrt{3}}\,\underline{i}}}$$

$$\omega_c BD = \omega a \tan 30^\circ = \frac{\omega a}{\sqrt{3}}$$

$$\therefore\ \omega_c = \frac{\omega a}{\sqrt{3}} . \frac{1}{2\sqrt{3}a} = \underline{\frac{\omega}{6} \circlearrowleft}$$

$$\underline{V}_c = \left(\frac{\omega a}{\sqrt{3}} . \frac{1}{2}\cos 60^\circ - \frac{2\omega a}{\sqrt{3}}\right)\underline{i} + \frac{\omega a}{\sqrt{3}} . \frac{1}{2}\sin 60\,\underline{j}$$

$$= \left(\frac{\omega a}{\sqrt{3}} . \frac{1}{2} . \frac{1}{2} - \frac{2\omega a}{\sqrt{3}}\right)\underline{i} + \frac{\omega a}{\sqrt{3}} . \frac{1}{2} . \frac{\sqrt{3}}{2}\,\underline{j}$$

$$\underline{\underline{\underline{V}_c = -\frac{7\sqrt{3}}{12}\omega a\,\underline{i} + \frac{\omega a}{4}\,\underline{j}}}$$

b) see acceleration diagram, measure directly:

$$\underline{a}_c = \omega^2 a . \frac{31\text{mm}}{100\text{mm}}\,\underline{i} + \omega^2 a \frac{43\text{mm}}{100\text{mm}}\,\underline{j}$$

$$\underline{\underline{\underline{a}_c = \omega^2 a\,(0.31\,\underline{i} + 0.43\,\underline{j})}}$$

$$\dot{\omega}_c\, BD = \omega^2 a . \frac{94\text{mm}}{100\text{mm}} = 0.94\,\omega^2 a$$

$$\dot{\omega}_c = \frac{0.94\,\omega^2 a}{2\sqrt{3}\,a} = \underline{\underline{0.27\,\omega^2}} \circlearrowright$$

$$\underline{a}_D = \omega^2 a \frac{11\text{mm}}{100\text{mm}}\,\underline{i} = \underline{\underline{0.11\,\omega^2 a\,\underline{i}}}$$

[9]

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alternatively, use trigonometry:

$$\underline{a}_C = \tfrac{1}{2}(\underline{a}_B + \underline{a}_D)$$

$$\underline{a}_B = \omega^2 AB . \cos 60\,\underline{i} + \omega^2 AB \sin 60\,\underline{j}$$

$$= \omega^2 a\left(\tfrac{1}{2}\,\underline{i} + \tfrac{\sqrt{3}}{2}\,\underline{j}\right)$$

$$\underline{a}_D = \frac{\omega_c^2\, BD}{\cos 30^\circ}\,\underline{i} = \frac{\omega^2}{36}\,2\sqrt{3}\,a\,\frac{2}{\sqrt{3}}\,\underline{i} = \frac{\omega^2 a}{9}\,\underline{i}$$

$$\underline{\underline{\simeq 0.11\,\omega^2 a\,\underline{i}}}$$

$$\therefore\ \underline{a}_C = \tfrac{1}{2}\omega^2 a\left[\left(\tfrac{1}{2} + \tfrac{1}{9}\right)\underline{i} + \tfrac{\sqrt{3}}{2}\,\underline{j}\right]$$

$$= \omega^2 a\left(\tfrac{11}{36}\,\underline{i} + \tfrac{\sqrt{3}}{4}\,\underline{j}\right)$$

$$\underline{\underline{\underline{a}_C = \omega^2 a\,(0.31\,\underline{i} + 0.43\,\underline{j})}} \quad (2.s.f.)$$

$$\mathring{\omega}_c \, BD = ab - be \tan 30^\circ$$

$$= \omega^2 AB - \omega_c^{\,2} BD \frac{1}{\sqrt{3}}$$

$$\mathring{\omega}_c = \frac{\omega^2 a - \frac{\omega^2}{36} \cdot 2\sqrt{3}a \cdot \frac{1}{\sqrt{3}}}{2\sqrt{3}a}$$

$$= \frac{\omega^2}{2\sqrt{3}} - \frac{\omega^2}{36\sqrt{3}} = \underline{\underline{\frac{\omega^2 17\sqrt{3}}{108} \circlearrowleft}} \simeq 0.27\omega^2$$

(2.s.f.)

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c) forces & torques acting on mechanism:

part	force or torque	velocity
piston (friction)	$+F$	$-\frac{2\omega a}{\sqrt{3}}$
piston.	$-m_p \frac{\omega^2 a}{9}$	$-\frac{2\omega a}{\sqrt{3}}$
con rod ↑	$-m_c \frac{\sqrt{3}}{4} \omega^2 a$	$+\frac{\omega a}{4}$
con rod →	$-m_c \frac{11}{36} \omega^2 a$	$-\frac{7\sqrt{3}}{12} \omega a$
con rod ↺	$-I \omega^2 \frac{17\sqrt{3}}{108}$	$+\frac{\omega}{6}$
crank.	T	ω

sum powers to zero to find T

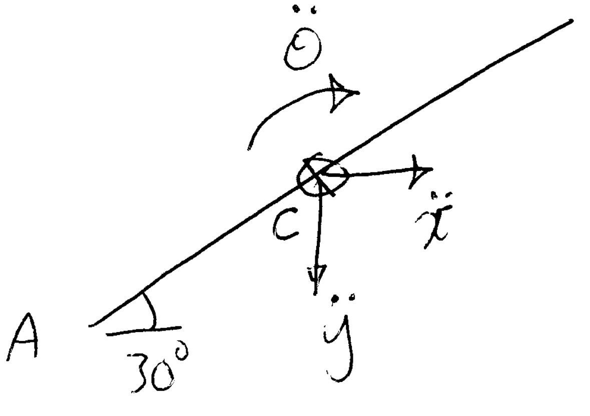
$$\underline{\underline{T = -\frac{1}{\cancel{\omega}}\left(-F.\frac{2\cancel{\omega}a}{\sqrt{3}} + m_p \frac{2\overset{2}{\cancel{\omega^3}}a^2}{9\sqrt{3}} - m_c \frac{\overset{2}{\cancel{\omega^3}}a^2\sqrt{3}}{16} + m_c \frac{\overset{2}{\cancel{\omega^3}}a^2 77\sqrt{3}}{432} - \frac{I\overset{2}{\cancel{\omega^3}} 17\sqrt{3}}{648}\right)}}$$

[10]

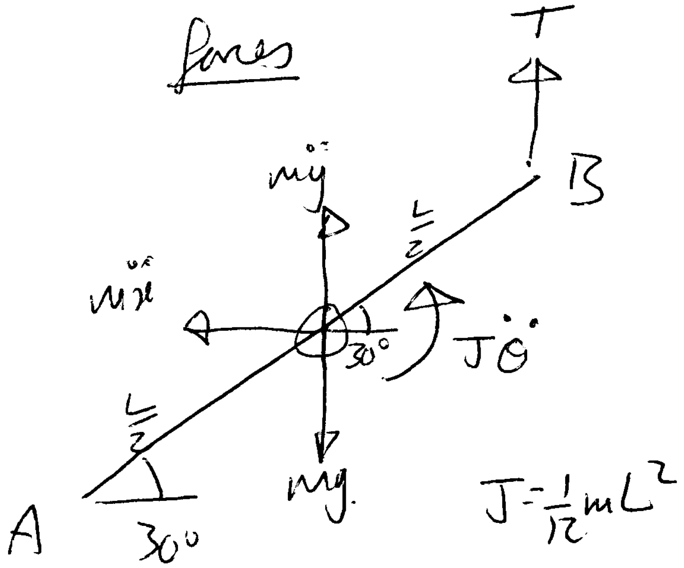
(a) Velocities. Nearly all solutions involved a velocity diagram. There was the usual occurrence of poorly annotated diagrams and absent signs/directions in the answers. (b) Accelerations. There were very few correct answers. Most solutions involved an acceleration diagram, but the Euler acceleration component of the connecting rod was often missing. (c) Virtual power. Very few correct answers, even accounting for errors carried over from (a) and (b). Errors included missing or incorrect terms, wrong signs, inclusion of gravity.

2.

a) accns



forces



sum forces → $m\ddot{x} = 0 \quad \therefore \quad \underline{\underline{\ddot{x} = 0}}$

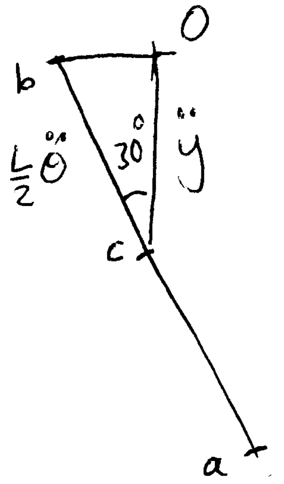
sum forces ↑ $m\ddot{y} + T = mg$ —— ①

mts about CoM $T \cdot \frac{L}{2} \cdot \frac{\sqrt{3}}{2} + J\ddot{\theta} = 0$

$$\therefore \quad T = -\frac{J\ddot{\theta}}{L} \frac{4}{\sqrt{3}}$$

$$T = -\frac{1}{12} mL^2 \frac{\ddot{\theta}}{L} \frac{L}{\sqrt{3}} = -\frac{mL\ddot{\theta}}{3\sqrt{3}}$$ (anti clockwise). ②

acceleration diagram to find $\ddot{y}$ in terms of $\ddot{\theta}$:



$$\ddot{y} = -\frac{L}{2}\ddot{\theta}\cos 30^\circ = -L\frac{\sqrt{3}}{4}\ddot{\theta}$$ ③

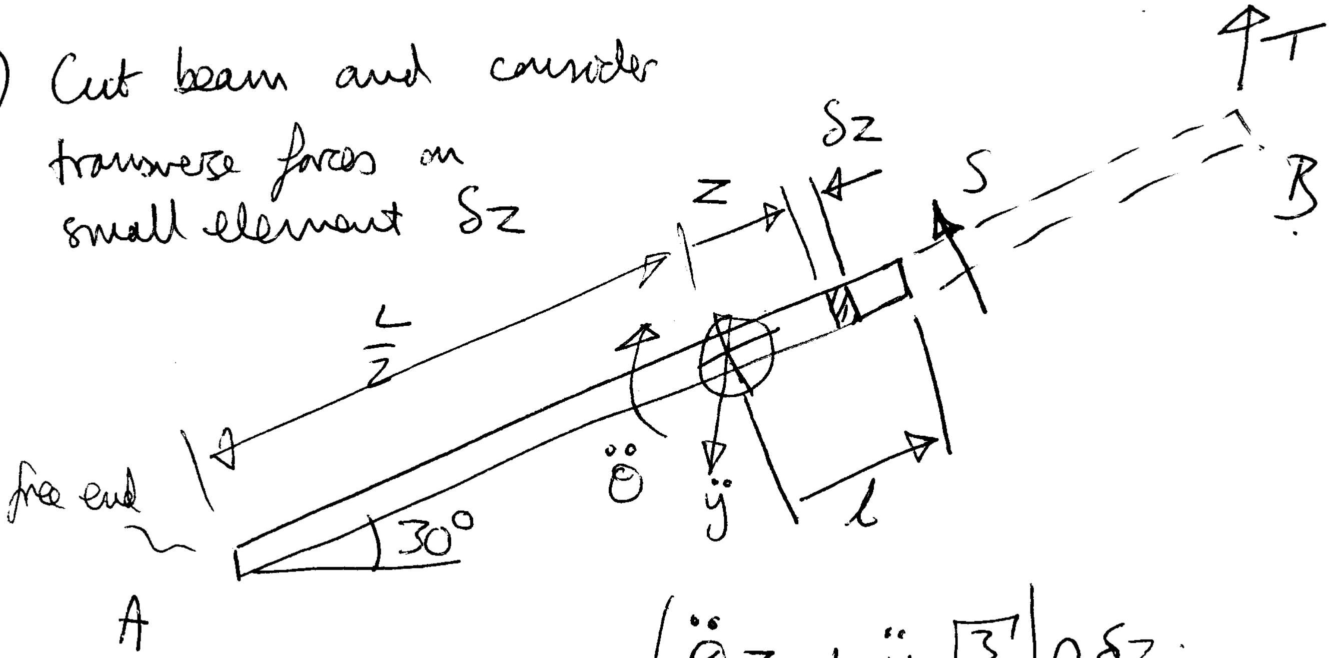
Subst ③ and ② into ①: $-mL\frac{\sqrt{3}}{4}\ddot{\theta} - \frac{mL\ddot{\theta}}{3\sqrt{3}} = mg$

$$\ddot{\theta} = \frac{-mg}{mL\frac{\sqrt{3}}{4} + \frac{mL}{3\sqrt{3}}} = \underline{\underline{-\frac{g}{L}\frac{12\sqrt{3}}{13}}}$$ [4]

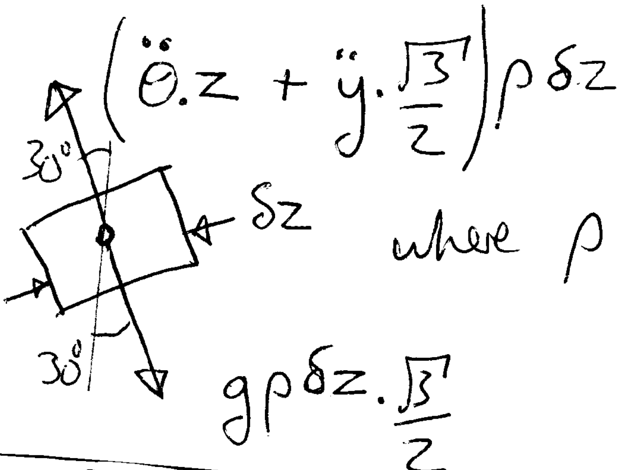
from ③: $\ddot{y} = L\frac{\sqrt{3}}{4}\frac{g}{L}\frac{12\sqrt{3}}{13} = \underline{\underline{g\frac{9}{13}}}$ [4]

from ①: $T = mg - m\ddot{y} = mg - mg\frac{9}{13} = \underline{\underline{mg\frac{4}{13}}}$ [4]

b) Cut beam and consider transverse forces on small element δz



D'Alembert and gravity forces on element:



where $\rho = \frac{m}{L}$

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hence load per unit length $w(z)\searrow = g\rho\frac{\sqrt{3}}{2} - \left(\ddot{\theta}z + \ddot{y}\frac{\sqrt{3}}{2}\right)\rho$

$$= \rho\left(g\frac{\sqrt{3}}{2} + \frac{g}{L}\frac{12\sqrt{3}}{13}z - g\frac{9}{13}\frac{\sqrt{3}}{2}\right)$$

$$= \underline{\underline{\rho g\frac{2\sqrt{3}}{13}\left(1 + 6\frac{z}{L}\right)}}$$

Sum forces perpendicular to cut beam A

$$S(l) - \int_{-\frac{L}{2}}^{l} w(z)\, dz = 0.$$

$$S(l) = \int_{-\frac{L}{2}}^{l} \rho g \frac{2\sqrt{3}}{13}\left(1 + 6\frac{z}{L}\right) dz$$

$$= \rho g \frac{2\sqrt{3}}{13}\left[z + \frac{3z^2}{L}\right]_{-\frac{L}{2}}^{l}$$

$$= \rho g \frac{2\sqrt{3}}{13}\left(l + \frac{3l^2}{L} + \frac{L}{2} - 3\frac{L}{4}\right)$$

$$\underline{\underline{S(l) = \rho g \frac{2\sqrt{3}}{13}\left(l + \frac{3l^2}{L} - \frac{1}{4}L\right)}}$$

check $S(l) = 0$ when $l = -\frac{L}{2}$ ✓

$S(l) = \rho g \frac{2\sqrt{3}}{13} L$ when $l = \frac{L}{2}$ ✓ $\left(= T\frac{\sqrt{3}}{2}\right)$

max BM when $S(l) = 0$

$$\frac{3l^2}{L} + l - \frac{1}{4}L = 0$$

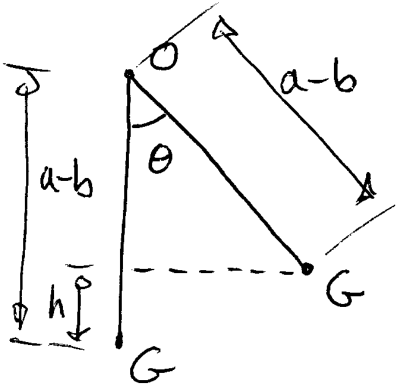
$$l = \frac{-1 \pm \sqrt{1 + 4 \cdot \frac{3}{L} \cdot \frac{1}{4}L}}{2 \cdot 3/L}$$

$l = \frac{L}{6}$ or $-\frac{L}{2}$ [13]

hence max BM at $l = L/6$, $\frac{2L}{3}$ from A

(a) Accelerations. Nearly all solutions incorrectly assumed that the beam rotates about B, with diagrams drawn showing centripetal, Euler and angular components of acceleration at the centre of mass. This assumption reduced the sytem from 3dof to 1dof and made the question much easier. (b) Shear force. Most solutions were successful in cutting the beam to introduce a shear force variable. The location of zero shear force was often found correctly.

3. a) PE of ring

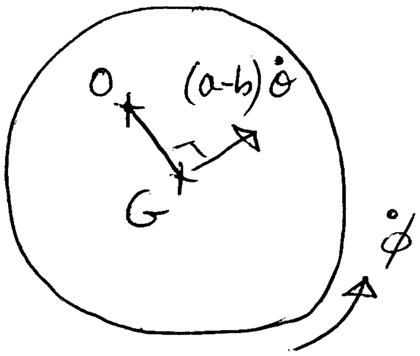


$$PE = mgh = mg\left((a-b)-(a-b)\cos\theta\right)$$

$$\underline{\underline{PE = mg(a-b)(1-\cos\theta)}}$$

[6]

b)



G rotates around O at $\dot{\theta}$

Ring rotates at $\dot{\phi}$

Find ϕ in terms of θ:

equal arc lengths $A'B = A''B$

$$\theta b = (\theta - \phi) a$$

$$\therefore \underline{\underline{\phi = \frac{\theta(a-b)}{a}}}$$

KE of ring

$$KE = \frac{1}{2}m(a-b)^2\dot{\theta}^2 + \frac{1}{2}J\dot{\phi}^2 \qquad J = ma^2$$

$$= \frac{1}{2}m(a-b)^2\dot{\theta}^2 + \frac{1}{2}ma^2\dot{\theta}^2\frac{(a-b)^2}{a^2}$$

$$\underline{\underline{KE = m\dot{\theta}^2(a-b)^2}}$$

[12]

c) max KE = max PE

$$m\dot{\theta}_{max}^2(a-b)^2 = mg(a-b)(1-\cos\theta_{max})$$

oscillation: $\dot{\theta}_{max} = \theta_{max}\omega$ small angles: $\cos\theta_{max} \simeq 1 - \frac{\theta_{max}^2}{2}$

$$m\theta_{max}^2\omega^2(a-b)^2 = mg(a-b)\frac{\theta_{max}^2}{2}$$

$$\therefore \omega^2 = \frac{g}{2(a-b)} \longrightarrow \underline{\underline{\omega = \sqrt{\frac{g}{2(a-b)}}}}$$

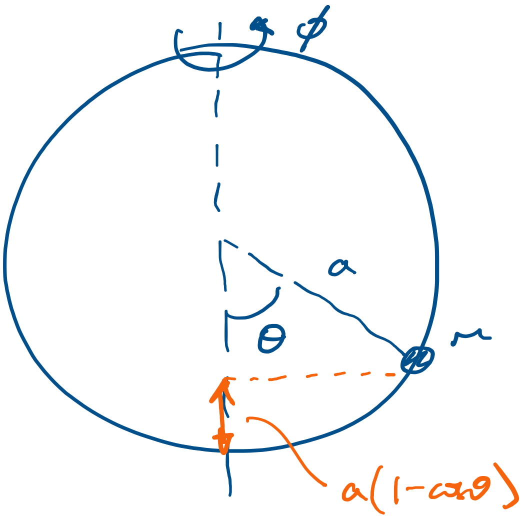
[7]

(a) Potential energy. Most solutions were correct. (b) Kinetic energy. Many solutions were incorrect due to errors in calculating the angular or translation velocities of the centre of the ring, or errors in the inertia of the ring. (c) Natural frequency. Most solutions used a correct method.

Very common across the solutions to all three questions were tiny or illegible diagrams and illegible writing.

Qu 4

(a)



$I_G = \frac{Ma^2}{2}$

$$T = \tfrac{1}{2}m\left(a^2\dot{\theta}^2 + a^2\sin^2\theta\,\dot{\phi}^2\right) + \tfrac{1}{2}\frac{Ma^2}{2}\dot{\phi}^2$$
$$= \tfrac{1}{2}ma^2\dot{\theta}^2 + \left(\tfrac{1}{2}ma^2\sin^2\theta + \tfrac{1}{4}Ma^2\right)\dot{\phi}^2$$

$V = "mgh" = mga(1-\cos\theta)$, redefine datum: $V = -mga\cos\theta$

so $L = T - V = \underbrace{\tfrac{1}{2}ma^2}_{\alpha}\dot{\theta}^2 + \left(\underbrace{\tfrac{1}{2}ma^2\sin^2\theta}_{f(\theta)} + \underbrace{\tfrac{1}{4}Ma^2}_{\beta}\right)\dot{\phi}^2 + \underbrace{mga}_{\gamma}\cos\theta$ //

(b) $\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} \longrightarrow \frac{d}{dt} = ma^2\ddot{\theta}$ //

$\frac{\partial L}{\partial \dot{\phi}} = \left(ma^2\sin^2\theta + \tfrac{1}{2}Ma^2\right).\dot{\phi} \longrightarrow \frac{d}{dt} = 2ma^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta + \left(ma^2\sin^2\theta + \tfrac{1}{2}Ma^2\right)\ddot{\phi}$ //

$\frac{\partial L}{\partial \theta} = ma^2\sin\theta\cos\theta\,\dot{\phi}^2 - mga\sin\theta$ //

$\frac{\partial L}{\partial \phi} = 0$ // $\longrightarrow$ note for later.

So: $\cancel{m}\cancel{a}^2\ddot{\theta} - \cancel{m}\cancel{a}^2\sin\theta\cos\theta\,\dot{\phi}^2 + \cancel{m}g\cancel{a}\sin\theta = 0$ —— ①

& $\left(\tfrac{1}{2}M\cancel{a^2} + m\cancel{a^2}\sin^2\theta\right)\ddot{\phi} + 2m\cancel{a^2}\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0$ —— ②

So: $a\ddot{\theta} + g\sin\theta - a\sin\theta\cos\theta\,\dot{\phi}^{2} = 0$ —— ①

& $\left(\frac{1}{2}M + M\sin^{2}\theta\right)\ddot{\phi} + 2M\dot{\theta}\dot{\phi}\sin\theta\cos\theta = 0$ —— ②

$\Big\}$ eom //

(c) $\frac{\partial L}{\partial \phi} = 0$ from (b)

hence $\frac{\partial L}{\partial \dot{\phi}} = \text{const} = \left(Ma^{2}\sin^{2}\theta + \frac{1}{2}Ma^{2}\right)\cdot\dot{\phi} \equiv C$ //

(d) from (c), $\dot{\phi} = \frac{C}{Ma^{2}\sin^{2}\theta + \frac{1}{2}Ma^{2}}$

Total energy: $E = \frac{1}{2}Ma^{2}\,\dot{}^{2} + \left(\frac{1}{2}Ma^{2}\sin^{2}\theta + \frac{1}{4}Ma^{2}\right)\dot{\phi}^{2} + mga(1-\cos\theta)$ //

$$E = \frac{1}{2}Ma^{2}\dot{\theta}^{2} + \frac{\left(\cancel{\frac{1}{2}Ma^{2}\sin^{2}\theta + \frac{1}{4}Ma^{2}}\right)C^{2}\cdot\frac{1}{2}}{\left(Ma^{2}\sin^{2}\theta + \frac{1}{2}Ma^{2}\right)^{\cancel{2}}} + mga(1-\cos\theta)$$

$$= \underbrace{\frac{1}{2}Ma^{2}\dot{\theta}^{2}}_{T} + \underbrace{\frac{C^{2}}{2\left(Ma^{2}\sin^{2}\theta + \frac{1}{2}Ma^{2}\right)} + mga(1-\cos\theta)}_{V_{eff}}$$

So $V_{eff} = -mga\cos\theta + \frac{C^{2}}{2Ma^{2}\sin^{2}\theta + Ma^{2}}$ //

with const chosen to be zero

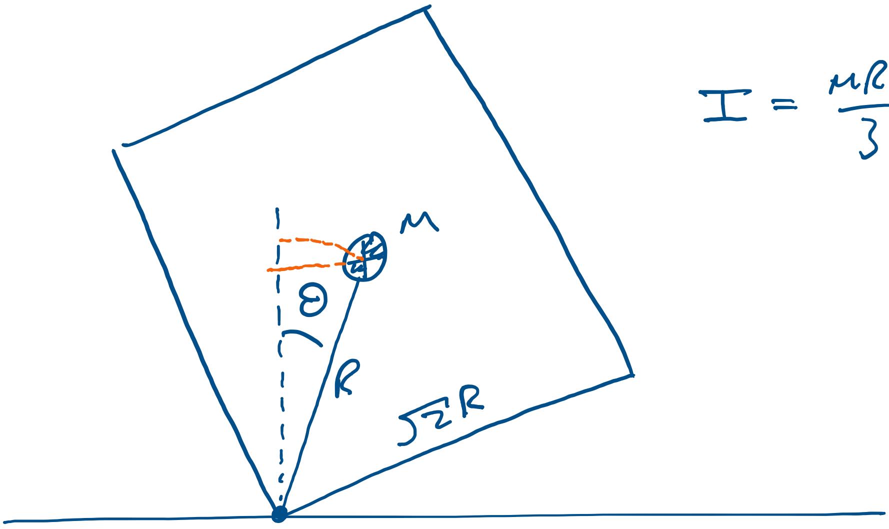
ie $B = mga$, $D = -2Ma^{2}$, $E = -Ma^{2}$ //

Note: incorrect -ve signs were not penalised for D & E as the question led candidates to expect +ve

(e) V_{eff} remains finite, so no infinite potential barriers, & bead can switch sides //.

A popular question. Most candidates correctly found the Lagrangian in Part (a). The equations of motion were accurately derived by most candidates in (b), and mistakes tended to be slips or inaccurate differentiation. Similarly the constant was clearly identified by most candidates in (c). Part (d) presented the most significant challenge, where many candidates struggled to derive the effective potential. In Part (e), few candidates identified that the lack of a singularity in the effective potential allowed the bead to travel to both halves of the hoop, contrasting with the example of the massless hoop in the lecture notes.

Qu 5 //



$$I = \frac{MR^2}{3}$$

(a)(i) $T = \frac{1}{2}M(\dot\theta R)^2 + \frac{1}{2}I\dot\theta^2$

$= \frac{1}{2}(I + MR^2)\dot\theta^2 \longrightarrow I = MR^2/3$

$= \frac{1}{2}\ \frac{4}{3}MR^2\ \dot\theta^2$

$= \underline{\frac{2}{3}MR^2\dot\theta^2}$, ie $\alpha = \frac{2}{3}MR^2$ //

$V = mgh = \underline{mg.R\cos\theta}$, ie $\beta = mgR$ //

(ii) $L = T - V = \underline{\frac{2}{3}MR^2\dot\theta^2 - mgR\cos\theta}$ //

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left(\frac{4}{3}MR^2\dot\theta\right) - mgR\sin\theta = 0.$$

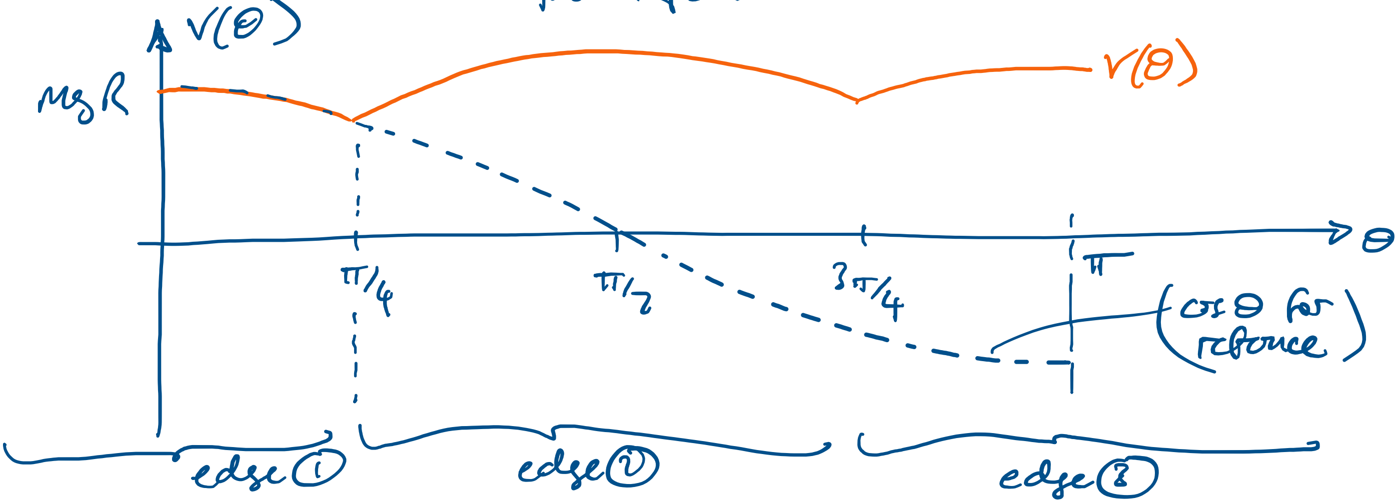
$$\frac{4}{3}\cancel{M}R^{\cancel{2}}\ddot\theta - \cancel{M}g\cancel{R}\sin\theta = 0.$$

$$\ddot\theta - \frac{3}{4R}g\sin\theta = 0$$ //

(iii) Valid for $-\pi/4 < \theta < \pi/4$, without edge switching //

(b)(i) $V(\theta) = mgR\cos\theta$

BUT $V(\theta)$ only valid for $-\pi/4 < \theta < \pi/4$.
then repeats

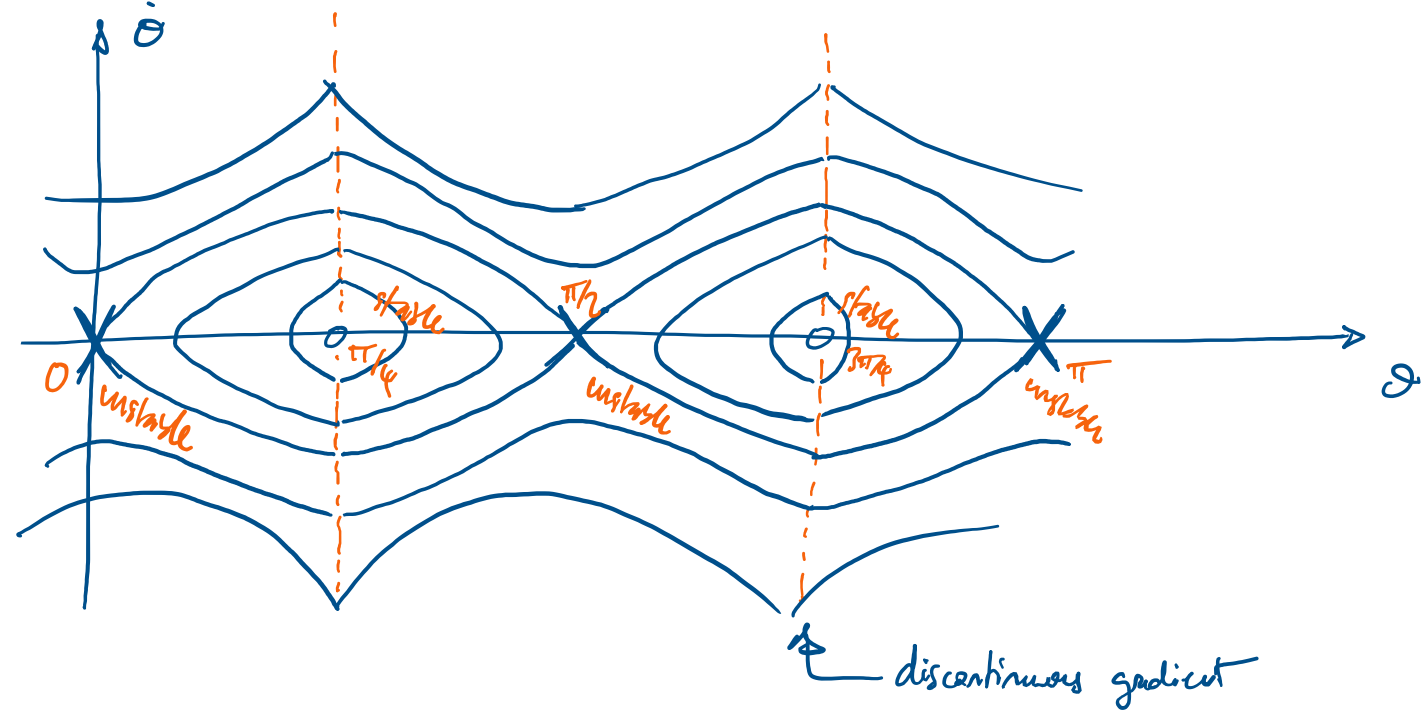


(ii) **unstable** equilibria @ $\theta_u = 0 + n\pi/2$

stable equilibria @ minima, ie $\theta_s = \pi/4 + n\pi/2$

V' discontinuous @ stable equilibria, but nevertheless the function is minimum @ θ_s

(iii) like pendulum, but truncated to range $-\pi/4 < \theta < \pi/4$ & periodic.



(c) $$\dot{\phi} = -D\sqrt{\sin\phi_0 + \cos\phi_0 - \sin\phi - \cos\phi}$$

small $\phi_0 \Rightarrow$ small ϕ

$$\text{so} \quad \dot{\phi} \simeq -D\sqrt{\phi_0 - \phi} = \frac{d\phi}{dt}$$

$$\int_{\phi_0}^{0} \frac{-1}{D\sqrt{\phi_0-\phi}}\,d\phi = \int_{0}^{T/4} dt$$

¼ period ie time for block to fall onto flat face & switch edges.

$$\left[\frac{2}{D}\sqrt{\phi_0-\phi}\right]_{\phi_0}^{0} = \frac{T}{4}$$

$$T = \frac{8}{D}\sqrt{\phi_0} \quad //$$

[Beyond scope: $\omega_n = \frac{2\pi}{T} \to \infty$ as $\phi_0 \to 0$

Natural frequency tends to infinity as release angle decreased.]

[Also beyond scope, note that in order to check whether the motion described in this question is even possible:

- The normal & tangential forces during rolling should be calculated to check that $\mu = F/N$ remains finite & N is +ve.
- The normal & tangential impulses during edge-switching should be calculated to check the same.
- This example highlights a danger with the Lagrangian approach as the forces are not calculated.]

Not a popular question, but those who attempted it generally did well. Part (a) presented few challenges, except that many candidates did not correctly identify the range of angles for which the equations were valid in (a)(iii). The potential energy in Part (b)(i) was generally sketched accurately, the most common mistake being to smooth out the discontinuities. Equilibria and stability were generally identified accurately in Part (b)(ii), but the phase portrait in (b)(iii) caused more difficulty, with most candidates missing the discontinuities. Hardly any candidates successfully integrated the given equation in (c) to find the period of oscillation: the most common errors were to integrate the differential equation with respect to phi directly rather than rearrange first; and to forget to multiply by four to account for a whole period.

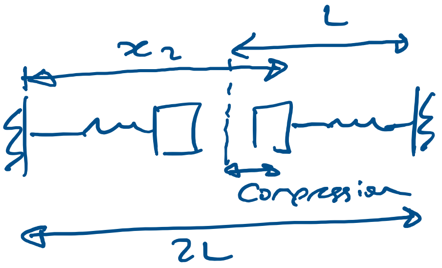
Qu6 //

(a) $V_{mag} = \frac{C}{x_2 - x_1}$

$V_{spring\,1} = \frac{1}{2}k(x_1 - a)^2$

$V_{spring\,2} = \frac{1}{2}k(x_2 - a)^2$

Spring 2 geometry:



$$\Rightarrow V = \frac{1}{2}k(a-x_1)^2 + \frac{1}{2}k(x_2-a)^2 + \frac{C}{x_2-x_1}$$

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$$

$$\Rightarrow L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(a-x_1)^2 - \frac{1}{2}k(x_2-a)^2 - \frac{C}{x_2-x_1} \quad //$$

ie $A = \frac{1}{2}m$, $B = \frac{1}{2}k$ //

as required

(b) $\frac{\partial V}{\partial x_1} = -k(a-x_1) + \frac{C}{(x_2-x_1)^2} = 0$ —— ①

$\frac{\partial V}{\partial x_2} = k(x_2-a) - \frac{C}{(x_2-x_1)^2} = 0$ —— ②

① + ② $\Rightarrow -\cancel{k}(a-x_1) + \cancel{k}(x_2-a) = 0$

$\Rightarrow x_1 + x_2 = 2a$

$x_2 = 2a - x_1$ // ie symmetric

$\rightarrow$ ① $\Rightarrow -k(a-x_1) + \frac{C}{((2a-x_1)-x_1)^2} = 0$

$$-\left[(2a-x_1)-x_1\right]^2 . k(a-x_1) + C = 0 .$$

$$-(2a-2x_1)^2 . k(a-x_1) + C = 0.$$

[Note expect $a > x_1$, so using $(a-x_1)$ not (x_1-a)]

$$-4k(a-x_1)^3 + C = 0$$

$$a - x_1 = \sqrt[3]{C/4k}$$

$$x_1 = a - \sqrt[3]{C/4k} \quad //$$

$$x_2 = a + \sqrt[3]{C/4k} \quad //$$

(c) consider $x_1 = x_{1eq} + x_1'$, $x_2 = x_{2eq} + x_2'$

$$x_1 = a/2 + x_1' \quad , \quad x_2 = 3a/2 + x_2'$$

$$L = \frac{1}{2}m\left(\dot{x}_1'^2 + \dot{x}_2'^2\right) - \frac{1}{2}k\left(a - a/2 - x_1'\right)^2 - \frac{1}{2}k\left(\frac{3a}{2} + x_2' - a\right)^2$$

$$- \frac{C}{3a/2 + x_2' - a/2 - x_1'}$$

$$L = \frac{1}{2}m\left(\dot{x}_1'^2 + \dot{x}_2'^2\right) - \frac{1}{2}k\left(a/2 - x_1'\right)^2 - \frac{1}{2}k\left(x_2' + a/2\right)^2 - \frac{C}{a + x_2' - x_1'}$$

need $$\frac{-C}{a + x_2' - x_1'} = \frac{-C}{a\left(1 + \frac{x_2' - x_1'}{a}\right)} \simeq -\frac{C}{a}\left(1 - \frac{x_2' - x_1'}{a} + \frac{\cancel{2}}{\cancel{2}}\frac{(x_2' - x_1')^2}{a^2}\right)$$

<u>Binomial to 2nd order</u>.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m\dot{x}_1 \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m\dot{x}_2$$

$$\frac{\partial L_{lin}}{\partial x_1'} = +k\left(a/_2 - x_1'\right) - \frac{c}{a^2} + \frac{2c}{a^3}\left(x_2' - x_1'\right)$$

$$\frac{\partial L_{lin}}{\partial x_2'} = -k\left(x_2' + a/_2\right) + \frac{c}{a^2} - \frac{2c}{a^3}\left(x_2' - x_1'\right)$$

Note that at equilibrium the const terms in $\frac{\partial L}{\partial q} = 0$,
So can ignore for equation of motion:

$$m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} + \begin{bmatrix} k + 2c/a^3 & -2c/a^3 \\ -2c/a^3 & k + 2c/a^3 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad //$$

Beyond qu scope: for this equilibrium $k = \frac{2c}{a^3}$
which gives the result in part (d)...
Candidates who derived this & answered with M, K from part (d) were given full marks.

(d) Find $k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \omega^2 m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$

$$k(2 - \alpha) = \omega^2 m \quad \text{——} \; ①$$

$$k(-1 + 2\alpha) = \omega^2 m\alpha \quad \text{——} \; ②$$

$$① \rightarrow ② \Rightarrow \cancel{k}(2\alpha - 1) = \alpha . \cancel{k}(2 - \alpha)$$

$$\cancel{2\alpha} - 1 = \cancel{2\alpha} - \alpha^2$$

$$\alpha^2 - 1 = 0 \Rightarrow \alpha = \pm 1$$

$$\& \quad \omega^2 = \frac{k}{m}(2 \mp 1)$$

Giving $\omega_1^2 = k/m$, $u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

& $\omega_2^2 = 3k/m$, $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ //

A popular question. Nearly all candidates correctly found the Lagrangian in (a), but the equilibria in (b) presented a much bigger challenge. Most candidates identified the simultaneous equations to solve, earning some of the available marks, but many then got lost in algebra rather than astute rearrangement. Attempts for Part (c) were rather mixed: many candidates correctly found the nonlinear equations of motion but could not linearise them, and many also did not spot that the constant terms would be zero for linearisation about an equilibrium. Although many candidates correctly answered Part (d), a significant number did not successfully find the eigenvalue/eigenvector decomposition.