EGT1
ENGINEERING TRIPOS PART IB

Monday 3 June 2024 9 to 11.10

Paper 1

MECHANICAL ENGINEERING

Answer not more than ***four*** *questions.*

Answer not more than **two** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

1 Figure 1(a) shows a T-shaped component that is pinned at its base to a fixed point at the origin O. The component is assembled using two uniform bars, each of mass m and length $2a$. Section OB remains orthogonal to Section ABC. The component is released from rest, with OB approximately vertical, and it subsequently starts to fall clockwise. The angle θ denotes the clockwise angular displacement of OB from vertical. Point X marks a general point along BC at a given distance x from B.

(a) Find the position of the centre of mass G of the assembled structure and show that the mass moment of inertia about G is $I_G = \frac{7}{6}ma^2$. [2]

(b) Find expressions for the position, velocity and acceleration vectors of Point X. Write your answer in terms of θ and its derivatives. [3]

(c) Find expressions for the angular velocity $\dot{\theta}$ and acceleration $\ddot{\theta}$ as functions of angular displacement θ. [6]

(d) Consider taking an imaginary cut at a distance x from B as illustrated in Fig. 1(b), in order to find the internal forces in the bar:

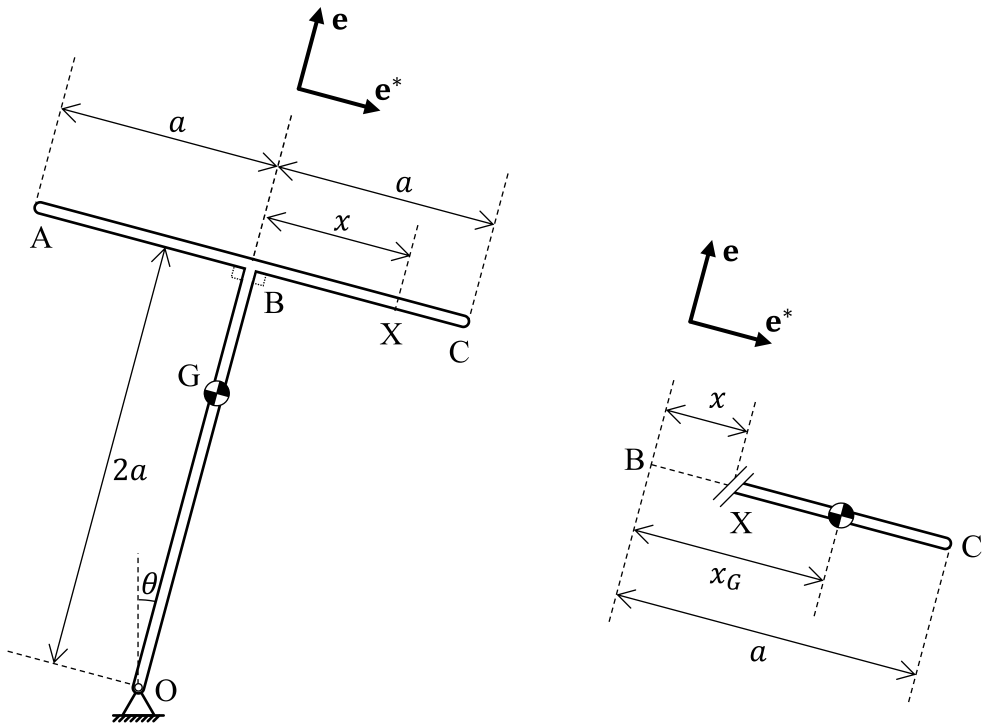
(i) Write down expressions for the mass m' of the cut section, and the distance x_G from B of its centre of mass. [2]

(ii) Find an expression for the internal shear force S. [8]

(e) When $\theta = 90°$, the expression for the shear force takes the form:

$$S \approx \lambda\,[2.12 + 0.265(1 + x/a)]$$

where λ represents the rest of the expression. Find the distance x from B for which the magnitude of the bending moment in Section BC reaches its maximum. [4]



(a) (b)

Fig. 1

2 The motion of a leg driving a pedal on a bicycle can be approximately modelled as a four-bar mechanism as illustrated in Fig. 2. Link AB is of length $L/2$, and links BC and CD are each of length L. The fixed pin joints shown at A and D represent the connections to the bike frame. In the position shown, AB is horizontal, BC is vertical, and CD is at an angle of 30° to the horizontal. Any effects from the other leg can be ignored throughout this question. AB rotates clockwise at constant angular velocity Ω.

(a) Identify the positions of the instantaneous centres for the three links AB, BC, and CD. [4]

(b) Find the angular velocity of BC and CD. [4]

(c) The mechanism is driven by a torque Q_C at Joint C. Friction torques Q_A and Q_B act at joints A and B resisting the motion. Find the input torque Q_C. [8]

(d) In addition to the friction torques, there is a drag force $F = \alpha v^2$, where v is the speed of the bicycle and α is a constant. The bike travels at constant speed v and AB remains at constant angular velocity Ω. Find the additional input torque ΔQ_C required due to the drag force. [4]

(e) Consider the case when the friction torque at A is Q_A, there is no friction in any other joint, and there is no drag. The lengths of BC and CD are both increased by an equal amount, and AB is not changed. Comment on how the drive torque Q_C would change, referring to the location of the instantaneous centre of BC and its angular velocity. [5]

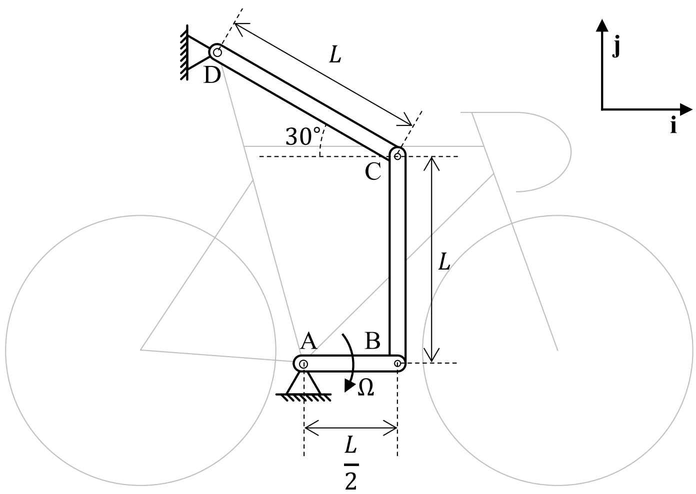


Fig. 2

3 Figure 3 shows a schematic of a floating wind turbine, which is tethered such that Point O remains fixed. The waves cause the wind turbine to rotate about Point O, such that the angle between the tower OA and vertical is $\theta(t)$. The tower OA has height h, and the turbine shaft AB has length d and is orthogonal to the tower. The blades of radius r rotate at constant speed ω (clockwise when facing the blades). The tip of one of the blades is labelled Point C and its angular displacement is ψ from the top position. The unit vectors $\mathbf{e}_1$, $\mathbf{e}_1^*$ are aligned to OA and AB, $\mathbf{j}$ is horizontal, $\mathbf{e}_2$ is aligned to BC, and $\mathbf{e}_2^* = \mathbf{e}_2 \times \mathbf{e}_1^*$.

(a) For the case when $\theta(t) = 0$:

(i) Find expressions for the time derivatives of the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_2^*$. [3]

(ii) Find the position, velocity and acceleration vectors $\mathbf{r}_C$, $\dot{\mathbf{r}}_C$, $\ddot{\mathbf{r}}_C$ of the blade tip C relative to O. Write your answer in terms of the given unit vectors. [4]

(b) For the case when $\theta(t)$ is time-varying:

(i) Find expressions for the time derivatives of the unit vectors $\mathbf{e}_1$, $\mathbf{e}_1^*$, $\mathbf{e}_2$ and $\mathbf{e}_2^*$. [6]

(ii) Find the position, velocity and acceleration vectors $\mathbf{r}_C$, $\dot{\mathbf{r}}_C$, $\ddot{\mathbf{r}}_C$ of the blade tip C relative to O, for a general rotation angle ψ. Write your answer in terms of the given unit vectors. [8]

(iii) Identify the Coriolis component of the acceleration at C and find its maximum possible magnitude, given that $\theta(t) = \Theta_0 \cos \Omega t$. [4]

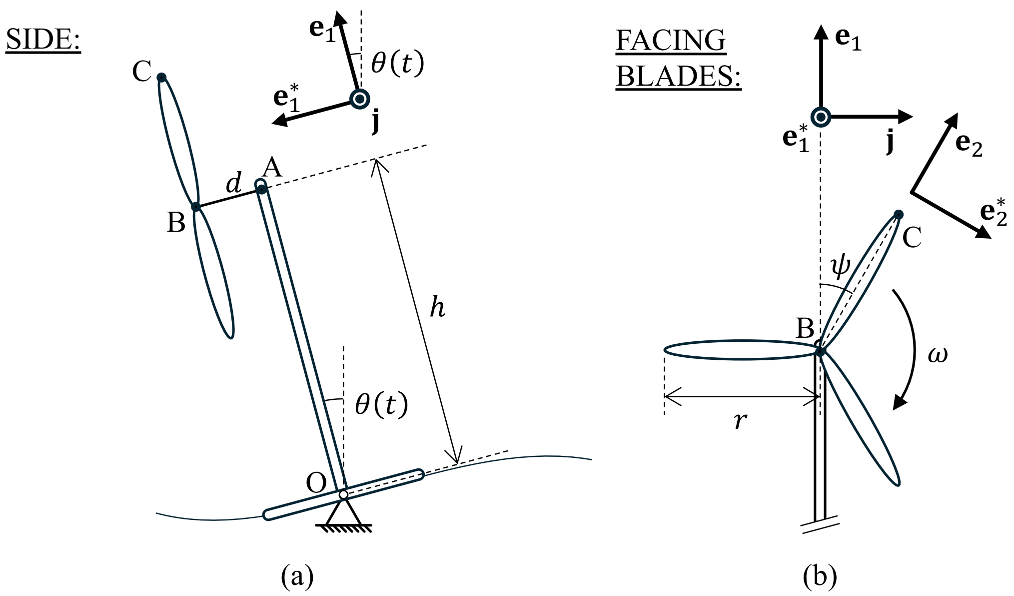


Fig. 3

SECTION B

4 A uniform rigid bar AB of mass m and length b is connected to a frictionless pivot at A and forms an angle θ with the vertical, as indicated in Fig. 4. AB is connected at B to a light frictionless slider, which links to a mass M via a horizontal spring CD of stiffness k. The position of the mass M is described by x, its absolute horizontal displacement from its equilibrium position when $\theta = 0$.

(a) Derive the Lagrangian of the system. [5]

(b) Use the Lagrangian to derive the equations of motion for x and θ. [5]

(c) Linearise the equations of motion about $x = 0, \theta = 0, \dot{x} = 0, \dot{\theta} = 0$. Show that they can be written in matrix form as:

$$\begin{bmatrix} M & 0 \\ 0 & \frac{mb^2}{3} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & -kb \\ -kb & kb^2 + mg\frac{b}{2} \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[5]

(d) Show that the natural frequencies of the system's normal modes can be expressed as:

$$\omega^2 = \frac{\beta \pm \sqrt{\beta^2 - 6m^2b^3kMg}}{2Mmb^2}$$

and give an expression for β. [7]

(e) Describe the motion corresponding to each normal mode. [3]

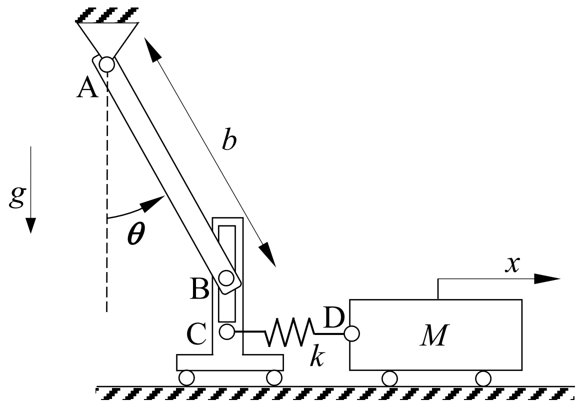


Fig. 4

5 A uniform rigid bar of mass M and length b is suspended by means of two springs AB and CD each of stiffness k, that can deform in the vertical direction. The springs are located at a distance c with respect to the centre of gravity G of the bar, as shown in Fig. 5. The motion of the bar can be described by ψ, the angle of the bar from horizontal, and y, the absolute downward vertical displacement of the centre of mass G from equilibrium. A mass m is mounted with a pin joint to G through a light inextensible link of length a, whose motion is described by the angle θ from vertical.

(a) Show that the Lagrangian of the system is given by:

$$L = \frac{1}{2}(M+m)\dot{y}^2 + \frac{1}{24}mb^2\dot{\psi}^2 + \frac{1}{2}m\left(a^2\dot{\theta}^2 - 2a\dot{y}\dot{\theta}\sin\theta\right)$$
$$-k(y^2 + c^2\sin^2\psi) + mga\cos\theta + E$$

where E is a constant offset. [6]

(b) Use the Lagrangian to find the equations of motion for y, θ, and ψ. [6]

(c) Linearise the equations of motion under the assumption of small amplitude vibration around $y = 0$, $\dot{y} = 0$, $\theta = 0$, $\dot{\theta} = 0$, $\psi = 0$, $\dot{\psi} = 0$. Show that they can be written in matrix form as:

$$\begin{bmatrix} m+M & 0 & 0 \\ 0 & ma^2 & 0 \\ 0 & 0 & \frac{mb^2}{12} \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} 2k & 0 & 0 \\ 0 & mga & 0 \\ 0 & 0 & 2kc^2 \end{bmatrix} \begin{bmatrix} y \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and comment on the resulting form of the normal modes. [6]

(d) The two springs are now replaced by two light inextensible rods, so that AB, BC, CD and AD form a four-bar mechanism. Sketch the four equilibrium positions for this new system. Without further calculation, comment on the stability of the four equilibrium positions found. [7]

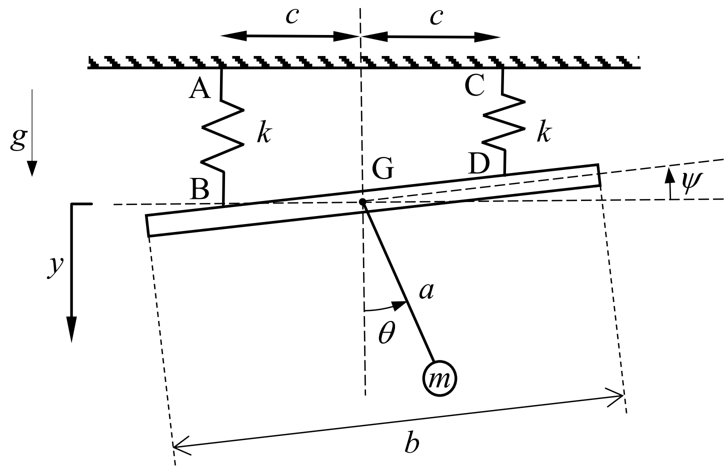


Fig. 5

6 Figure 6 shows a point mass m moving under gravity. The mass is connected by a spring of stiffness k to a fixed origin O. The position of the mass is given by polar coordinates r, θ, and ϕ. The spring force acts in the radial direction and is zero when $r = 0$.

(a) Initially, the system is driven at a constant rotation rate $\dot{\phi} = \Omega$, and θ is constrained to a fixed angle $\theta = \theta_0$.

(i) Find the Lagrangian for the system $L(r, \dot{r})$. [4]

(ii) Derive the equation of motion for r. [4]

(iii) Sketch the phase portraits of the system when $\Omega^2 < k/(m \sin^2 \theta_0)$ and $\Omega^2 > k/(m \sin^2 \theta_0)$, labelling the position of any equilibria. [7]

(b) The spring is now connected to the origin O with a frictionless pivot, so that the angular coordinates are now no longer constrained and the system has three degrees of freedom.

(i) Derive the new Lagrangian for the system $L(r, \dot{r}, \theta, \dot{\theta}, \phi, \dot{\phi})$. [4]

(ii) Use the Lagrangian to demonstrate that the angular momentum about the vertical axis through O is conserved. [4]

(iii) Consider starting the mass m from two different locations that differ only by the value of ϕ. What can you say about the two motions? [2]

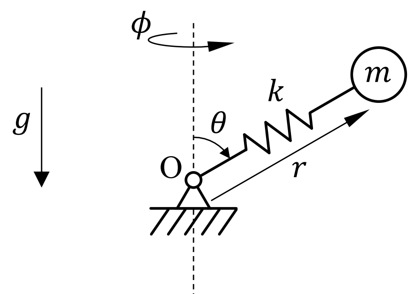


Fig. 6

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