2P6 - Crib 2022 - A gv, B sjg)

2P6 2022 Crib

1) a) i) System needs to be asymptotically stable. (4)
If so, input sinusoids at different frequencies and measure gain and phase shift of response. Response may be large, and take a long time to settle down, near resonance.

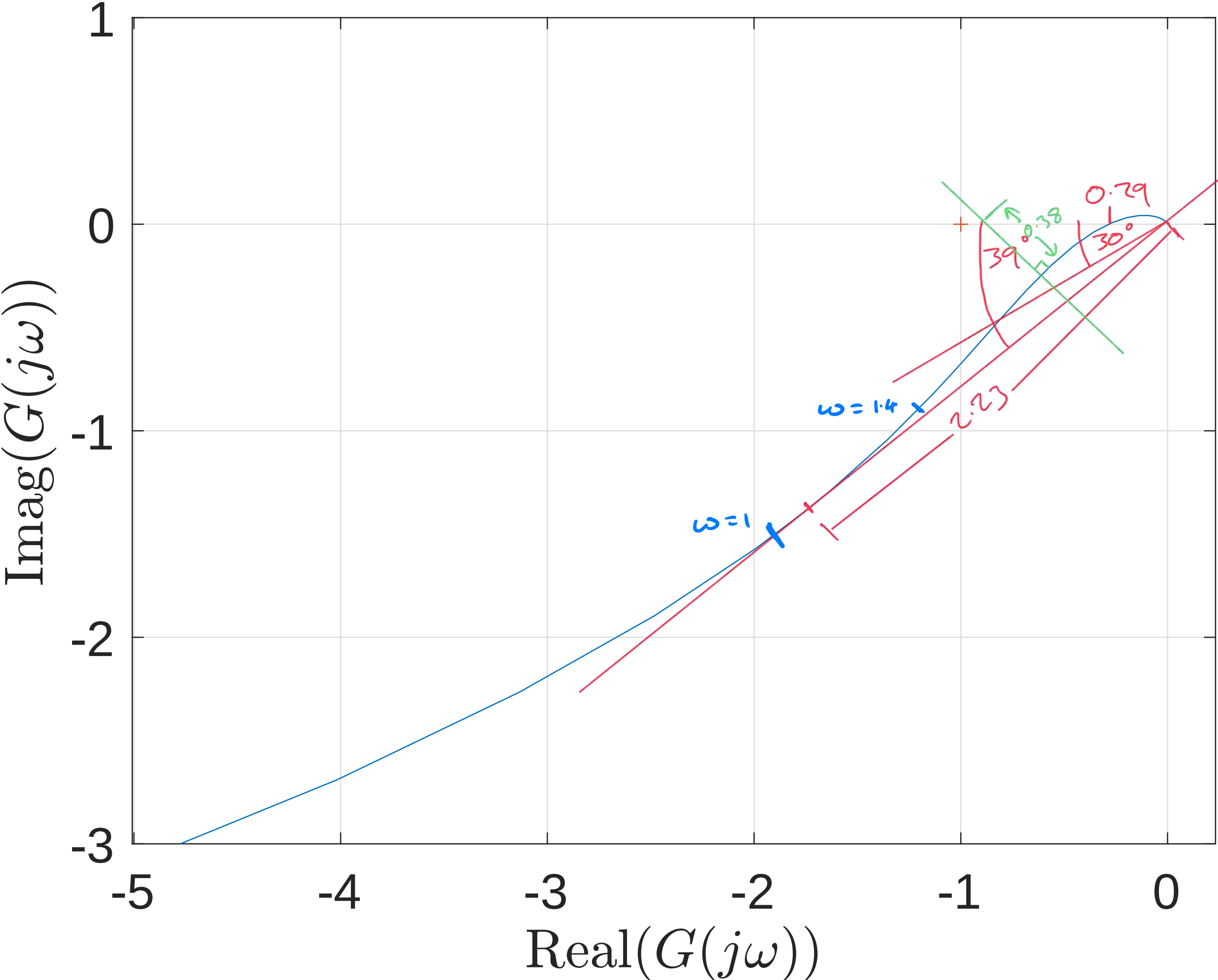
ii) The closed loop system is stable if and only if the Nyquist diagram of the open loop transfer function ($T_{u\to y}(j\omega)$) leaves the $-\frac{1}{k}$ point to its left (or does not encircle -1) (4)

b) i) Phase margin $= 30°$
gain margin $= 1/0.29 = 3.45$
$\Rightarrow$ If gain of real system is up to 3.45 greater OR phase lag up to 30° more lag then closed loop system is stable (6)

ii) PM $= 39°$ when $k = 1/2.23 = 0.45$
$\omega \approx 1.1$

$$\omega\tau = \frac{39}{180} \times \pi \Rightarrow \tau = 0.62$$ (6)

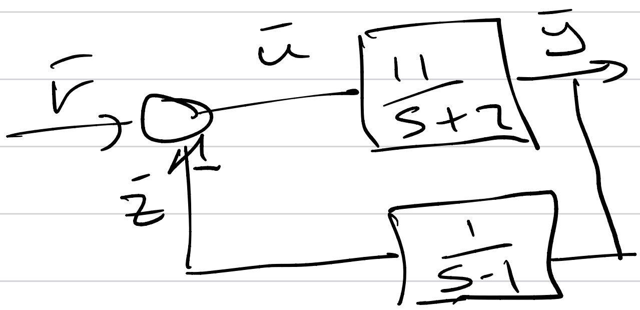
iii) $\max |1/1+L| = 1/0.38 = 2.63$ (5)



1
0
-1
-2
-3
-5
-4
-3
-2
-1
0
Real(G(jω))
Imag(G(jω))
0.29
0.38
39°
30°
2.23
ω = 1.4
ω = 1

2)

a)



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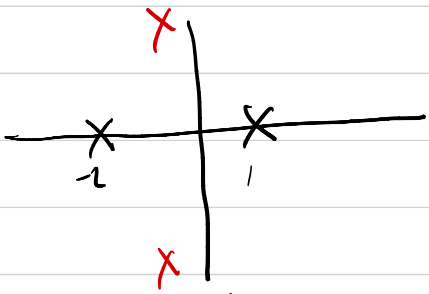
b)



x - OL poles

x - CL poles

CLCE $\quad 1+\frac{11}{(s-1)(s+2)}=0 \iff (s-1)(s+2)+11=0$

$\iff s^2+s+9=0$

$\iff s=-\frac{1}{2}\pm\sqrt{8.75}\,j$

OL is unstable

CL is asymptotically stable.

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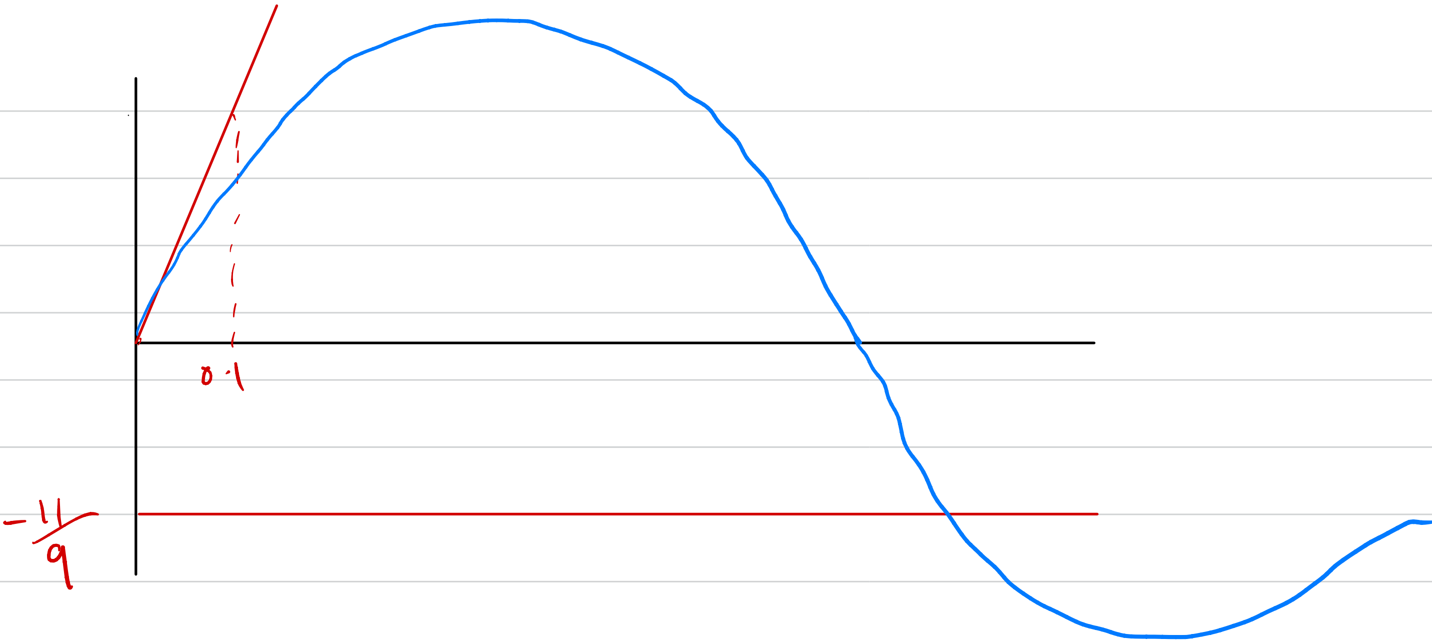
c) $\bar y(s) = \dfrac{11/(s+2)}{1+\frac{11}{(s+2)(s-1)}}\,\bar r(s) = \dfrac{11(s-1)}{s^2+s+9}\,\bar r(s)$

$\bar u(s) = \dfrac{1}{1+\frac{11}{(s+2)(s-1)}} = \dfrac{(s+2)(s-1)}{s^2+s+9}\,\bar r(s)$

$G_1 = T_{\bar r\to\bar y}$: $G_1(0) = \dfrac{-11}{9}$ = final value

initial slope $= \lim_{s\to\infty} s\,G(s) = 11$

$\omega_n=3$, $2\zeta\omega_n = 1 \Rightarrow \zeta = 1/6 = 16\%$

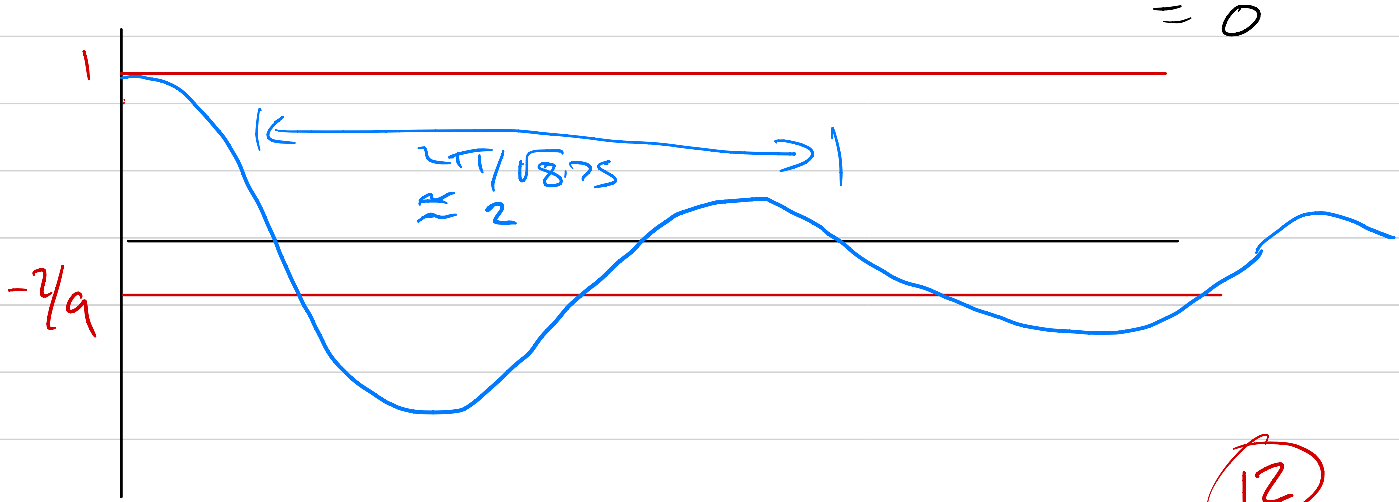


$$G_n = T_{r\to u} = \frac{(s^2+s-2)}{(s^2+s+9)} = 1 \quad \frac{-11}{s^2+s+9}$$

Initial value $= 1$

Final Value $= -2/9$

Initial slope $= 0$



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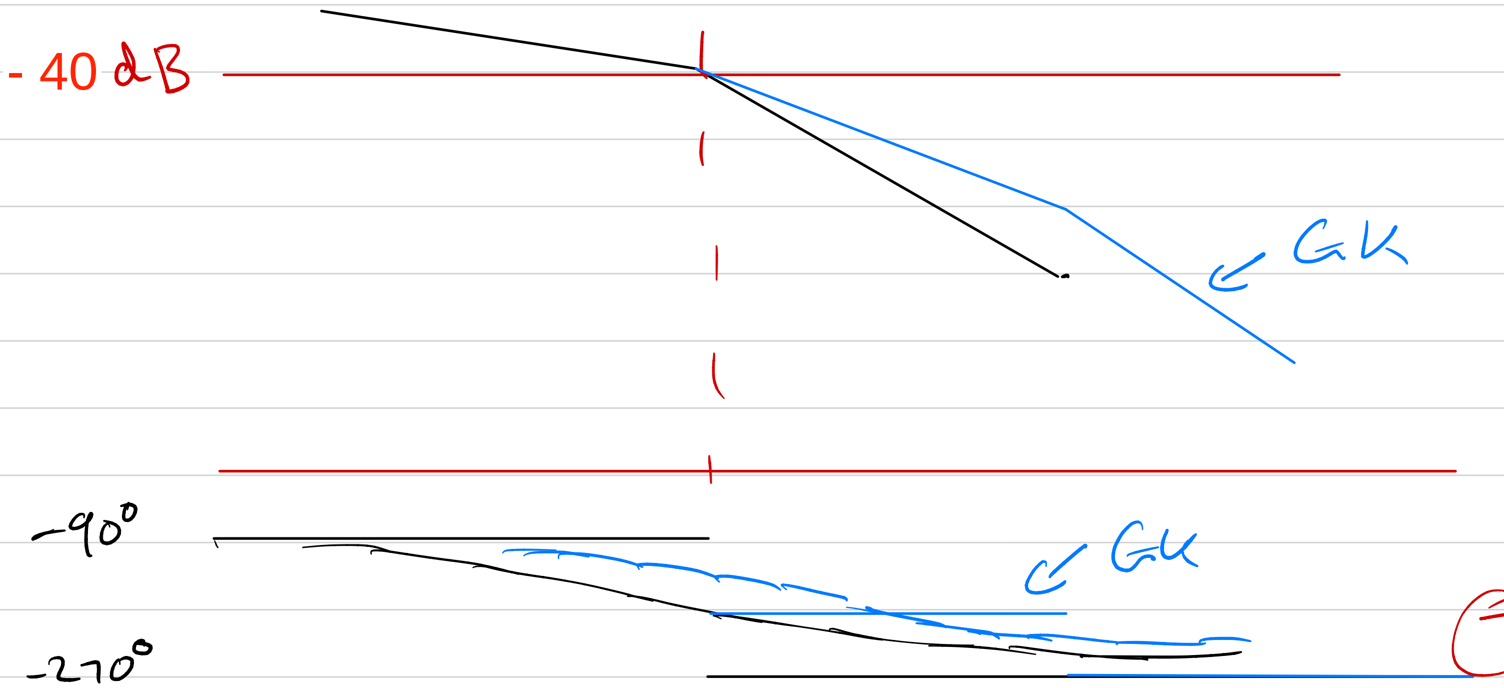
3) a) $G(s) = \dfrac{10}{s(s+10)^2} = \dfrac{0.1}{s(0.1s+1)^2}$



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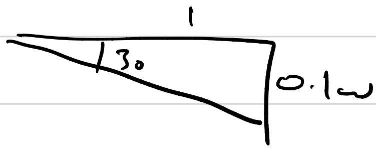
b) $\angle G(j\omega_c) = -150^\circ = -90^\circ + 2\,\mathrm{atan}\,0.1\omega$

$\Rightarrow \mathrm{atan}\,0.1\omega = 30^\circ$

$\Rightarrow 0.1\omega = \tan 30^\circ = \sqrt{3}/3$

$\Rightarrow \omega = 10/\sqrt{3} = 5.78$

$|G(j\omega_c)| = \dfrac{0.1}{\frac{10}{\sqrt{3}} \cdot \left(1 + \left(\frac{1}{\sqrt{3}}\right)^2\right)}$



$= \cancel{0.0027}\ 0.013$

$\Rightarrow K = 1/\cdot = \cancel{435}\ \underline{77}$

$\angle G(10j) = -180^\circ$, $|G(j10)| = \dfrac{0.1}{10 \cdot 20}$

$\Rightarrow GM = \cancel{2.175}\ \underline{2.6}$

BW $\approx$ $\underline{10 \text{ rad/s}}$

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c) $GK = K_1 \frac{10}{s(s+10)} \times \frac{10 \cancel{(s+10)}}{s+100} = \frac{K_1 100}{s(s+10)(s+100)}$

See above for Bode (in blue) 5

d) Greater bandwidth $\Rightarrow$ faster response

Same PM $\Rightarrow$ Similar damping ratio 5

4.(a) Suppose that a signal $f(t)$ has Fourier transform $F(\omega)$, defined as follows:

$$F(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{otherwise} \end{cases}$$

Determine the time function $f(t)$. [3]

Solution: *Using inverse Fourier transform:*

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_0}^{+\omega} e^{j\omega t}d\omega \\ &= \frac{1}{2j\pi t}[e^{j\omega t}]_{-\omega_0}^{+\omega} \\ &= \frac{1}{2j\pi t}(2j\sin(\omega_0 t)) = \frac{\sin(\omega_0 t)}{\pi t} \end{aligned}$$

Solutions using tables and the duality theorem would also be acceptable

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What is the total energy of $f(t)$? [2]

Solution: *Using Parseval to compute it in the frequency domain:*

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega = \omega_0/\pi$$

(b) A new signal is made up as the product of functions as follows:

$$f_1(t) = \frac{\sin(\omega_1 t)}{t}\frac{\sin(\omega_2 t)}{t}$$

where $\omega_1 < \omega_2$. Determine the Fourier transform of $f_1(t)$ and sketch it as a function of frequency. [8]

Solution: *From the first part, or tables, the FT of each component is*

$$F_1(\omega) = \begin{cases} \pi, & |\omega| < \omega_1 \\ 0, & \textit{otherwise} \end{cases}$$

and

$$F_2(\omega) = \begin{cases} \pi, & |\omega| < \omega_2 \\ 0, & \textit{otherwise} \end{cases}$$

The convolution theorem states that we need $F_1 * F_2/2\pi$:

$$F_1 * F_2/2\pi = \begin{cases} \omega_1\pi, & |\omega| < \omega_2 - \omega_1 \\ \omega_1\pi(1 - (|\omega| - (\omega_2 - \omega_1))/(2\omega_1)), & \omega_2 - \omega_1 \le |\omega| \le \omega_1 + \omega_2 \\ 0, & \textit{otherwise} \end{cases}$$

Sketch:... flat central section from 0 up to $\omega_2 - \omega_1$ and then linear decay to zero up to $\omega_2 + \omega_1$.

(c) Determine the fraction of the total energy in $f_1(t)$ that lies above frequency $\omega_2 - \omega_1$. [7]

Solution: *We use Parseval to find the energy within each frequency range (noting that we must include negative and positive frequencies at each ω, hence factors '$2\times$'):*

0 *to* $\omega_2 - \omega_1$*:*

$$2 \times (1/(2\pi))\omega_1^2\pi^2(\omega_2 - \omega_1)$$

$+\omega_2 - \omega_1$ *to* $+\omega_2 + \omega_1$*:*

$$2 \times 1/(2\pi)\omega_1^2\pi^2 \int_0^{2\omega_1} \omega^2/(4\omega_1^2)d\omega \quad \textit{(using change of variables } \omega \to (\omega_2 + \omega_1 - \omega))$$
$$= \pi/12[\omega^3]_0^{2\omega_1} = 2/3\pi\omega_1^3$$

And so ratio is:

$$\frac{2/3\pi\omega_1^3}{2/3\pi\omega_1^3 + 2 \times (1/(2\pi))\omega_1^2\pi^2(\omega_2 - \omega_1)} = \frac{2\omega_1}{(3\omega_2 - \omega_1)}$$

(d) It is desired to sample the signal at regular times, obtaining samples $f_1(nT)$. Make two sketches of the spectrum of the sampled signal, corresponding to the two cases $T = \pi/(\omega_2)$ and $T = \pi/(\omega_2 + \omega_1)$. What is the maximum allowable sampling period T if $f_1(t)$ is to be perfectly reconstructed from its sampled values? [5]

Solution: $T = \pi/(\omega_2)$*:*

In this case sampling frequency is $2\omega_2$, which is not twice the max. frequency component in f_1, so aliasing distortion occurs.

Sketch - sampled spectrum is a flat line across all frequencies.

$T = \pi/(\omega_2 + \omega_1)$*:*

Now we have $2\omega_2$, which is twice the max. frequency component in f_1, so no aliasing distortion occurs and the spectra are adjacent to each other without overlap:

Sketch ...

Hence $T = \pi/(\omega_2 + \omega_1)$ *is the maximum allowable sampling period.*

END OF PAPER

5)

a) $y(t) = m(t)^2 + 2m(t)\cos 2\pi f_c t + \cos^2 2\pi f_c t.$

$$= \underbrace{2m(t)\cos(2\pi f_c t)}_{\text{want to keep this}} + \underbrace{m(t)^2 + 1 + \cos 4\pi f_c t}_{\text{want the filter to reject these}}.$$

want to keep this

(spectrum btw $(f_c - w, f_c + w)$
$(-f_c - w, -f_c + w)$

want the filter to reject these.

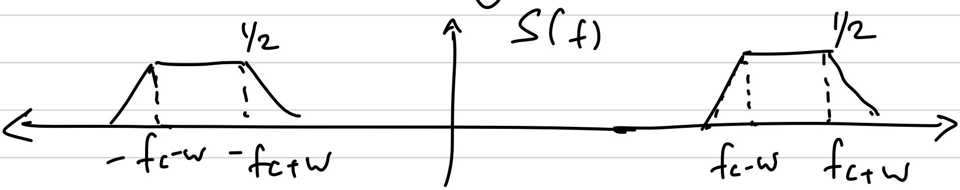
$m(t) \rightarrow$ spectrum from $-w$ to w.

$m^2(t) \Leftrightarrow M(f) * M(f) \rightarrow$ spectrum from $-2w$ to $2w$.

$1 \longrightarrow \delta(f)$

$\cos 4\pi f_c t \longrightarrow \frac{1}{2}\left(\delta(f - 2f_c) + \delta(f + 2f_c)\right)$

If we apply a bandpass filter:



$y(t) * S(t) = m(t)\cos(2\pi f_c t)$

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b) i) Optimal detection rule:

If $\hat{x} = (\hat{x}_1, \dots \hat{x}_7)$ is the detected seq.: for $i = 1, \dots 7$:

$$\hat{x}_i = \begin{cases} 2 & \text{if } \hat{y}_i \geq 0 \\ -2 & \text{if } \hat{y}_i < 0. \end{cases}$$

$$\hat{r}_i = \begin{cases} 0 & \text{if } \hat{y}_i \geq 0 \\ 1 & \text{if } \hat{y}_i < 0. \end{cases}$$

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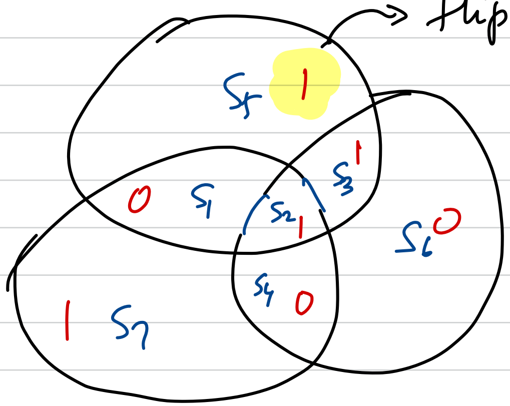
(ii) Sequence of received bits.

$\hat{r} = [0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1]$



→ flip $\hat{r}_5$ so that all circles have even parity.

Therefore decoded Hamming codeword

$\hat{c} = [0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1]$

Transmitted info bits = 0110.

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iii) A decoding error occurs with a Hamming code if the channel flips two or more bits.

Prob. of decoding error

$= 1 -$ Prob. (channel flips 0 or 1 bits)

$= 1 - (0.8)^7 - 7(0.8)^6(0.2)$

$= \underline{\underline{0.4233}}$

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6)

(a) The DFT for a signal is defined as

$$X_k = \sum_{n=0}^{N-1} x_n \exp(-j2\pi nk/N)$$

(i) If x_n regularly sampled every 3 s from a continuous-time function $x(t)$, what frequency in Hz would component X_3 correspond to when $N = 10$? [3]

Solution: *Sampling frequency is* $2\pi/3\ rad.s^{-1}$. *Split this into* $N = 10$ *bins of width* $2\pi/30\ rad.s^{-1}$. *Hence* X_3 *corresponds to* $3 \times 2\pi/30 = 2\pi/10\ rad.s^{-1}$, *i.e.* $0.1 Hz$.

(ii) Show that the DFT of a constant signal $x_n = C$ is given by

$$X_k = \begin{cases} CN, & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

[5]

Solution:

$$\begin{aligned} X_k &= C \sum_{n=0}^{N-1} \exp(-j2\pi nk/N) \\ &= \frac{(1 - \exp(-j2\pi k)}{(1 - \exp(-j2\pi k/N)} \\ &= \frac{0}{(1 - \exp(-j2\pi k/N)} \end{aligned}$$

Now, numerator always zero, but could denominator also be zero (and hence indeterminate). For $k = mN$, *for integer* m, *denominator would be zero. Hence consider this case as a limit, or just rework as a special case for* $k = 0$ *(i.e.* $m = 0$*):*

$$\begin{aligned} X_k &= C \sum_{n=0}^{N-1} \exp(-j2\pi n0/N) \\ &= CN \end{aligned}$$

Hence DFT is as proposed.

(iii) The inverse DFT is defined as

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \exp(j2\pi nk/N)$$

 (cont.

Show that use of this formula to calculate x_n for n outside the range 0 to $N-1$ implies periodicity of the data, i.e.

$$x_{n+N} = x_n$$

and, for the case where X_k is purely imaginary,

$$x_{-n} = -x_n^*$$

[5]

Solution:

$$\begin{aligned} x_{n+N} &= \frac{1}{N}\sum_{k=0}^{N-1} X_k \exp(j2\pi(n+N)k/N) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X_k \exp(j2\pi nk/N) = x_n \end{aligned}$$

so data can be considered 'periodic'.
When X_k is purely imaginary:

$$X_k = jX_k^{(i)}$$

And so,

$$\begin{aligned} x_{-n} &= \frac{j}{N}\sum_{k=0}^{N-1} X_k^{(i)} \exp(+j2\pi nk/N) \\ &= \frac{-1}{N}\left(\sum_{k=0}^{N-1} jX_k^{(i)} \exp(-j2\pi nk/N)\right)^* \\ &= -x_n^* \end{aligned}$$

as required.

(b) The Fourier transform of a baseband signal $x_b(t)$ is shown in Fig. 1. A passband signal $x(t)$ is generated from $x_b(t)$ and a complex sinusoid of frequency f_c as follows.

$$x(t) = \mathrm{Re}\left[x_b(t)e^{j2\pi f_c t}\right].$$

Here $\mathrm{Re}(a)$ denotes the real part of the complex number a.

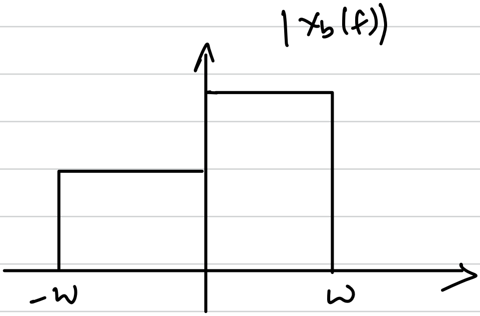
(i) Show that the Fourier transform of $x_b^*(t)$ is $X_b^*(-f)$. (Here a^* denotes the conjugate of a complex number a.) [3]

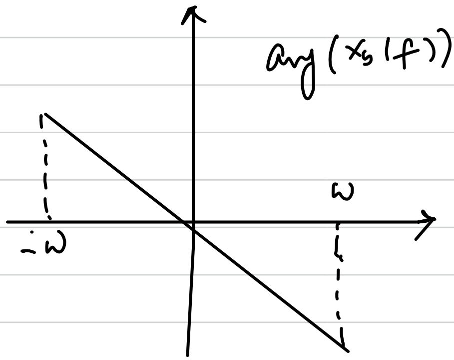
(ii) Determine an expression for $X(f)$, the Fourier transform of $x(t)$, in terms of $X_b(f)$. *Hint*: For a complex number a, $\mathrm{Re}(a) = \frac{a+a^*}{2}$. [4]

(iii) Sketch $X(f)$. [5]

2P6 Comms. Solutions - 2022.

6) (b)





(i) $FT(x_b^*(t))$

$$= \int x_b^*(t)\, e^{-j2\pi f t}\, dt.$$

$$= \left(\int x_b(t)\, e^{j2\pi f t}\, dt \right)^*$$

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$$= X_b^*(-f)$$

(ii) $$x(t) = \frac{1}{2}\left[x_b(t)\, e^{j2\pi f_c t} + x_b^*(t)\, e^{-j2\pi f_c t} \right]$$

$$\Rightarrow \quad X(f) = \frac{1}{2}\left[X_b(f-f_c) + X_b^*(-(f+f_c)) \right]$$

(using part (i) and freq. shift property of FT.)

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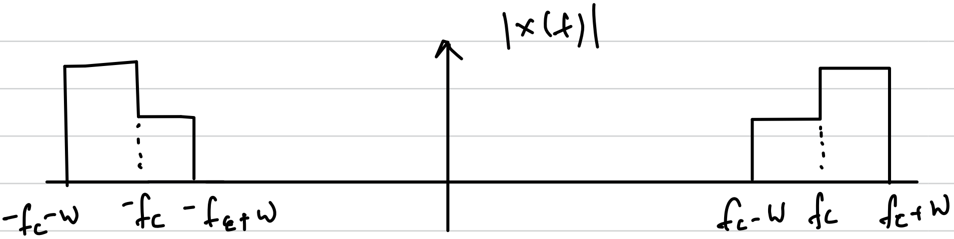
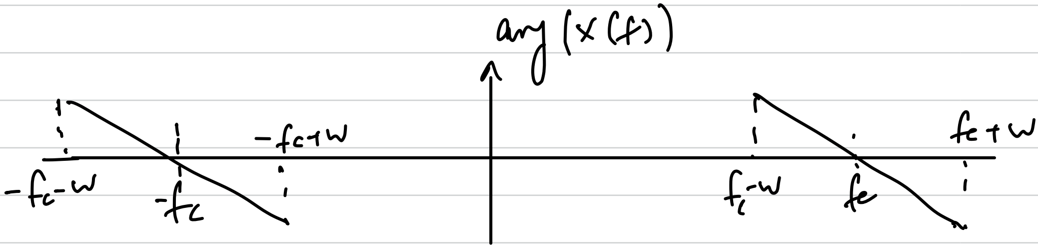
iii) since $|X_b^*(-f-f_c)| = |X_b(-f-f_c)|$

and $\arg(X_b^*(-f-f_c)) = -\arg(X_b(-f-f_c))$

we have:





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