2P6 Solutions 2024

SECTION A

1. (a) To carry out an experimental procedure to find the Bode diagram of a physical system it must be linear and time-invariant system. Also it must be stable, unless the system is equipped with a stabilising controller. The procedure is to input sine waves $A\sin(\omega t)$ and to measure the steady-state output which must equal $|G(j\omega)|A\sin(\omega t - \angle(G(j\omega)))$. Measuring the gain and phase shift allows the magnitude and phase for a frequency to be determined and to be marked on a Bode diagram. Repeating this experiment for a range of frequencies $\omega_1, \omega_2, \ldots$ allows a Bode diagram to be constructed. [6]

   (b) (i) A straight-line asymptote at low frequencies has slope -20dB/dec and passes through $\omega = 1$ suggesting $ac/(d_1 d_2) = 1$. The slope decreases further with a break point at 0.4 rad/sec (where the phase is $-135°$) which suggests $d_1 = 0.4$. The notch (anti-resonance) at around 12 rad/sec suggests $c = 144$. The corresponding rapid drop in phase at the notch (in contrast to the expected rise) suggests that the associated zeros are in the right half-plane rather than the left half-plane, i.e. b is negative. From the plots in the mechanics data book the damping factor might be around 0.1, hence $b = -2$ seems a plausible value. Without any further poles or zeros the magnitude would flatten, but it starts to roll off again suggesting another pole, i.e. $d_2 = 80$ (where the phase is $-405°$). Hence the values:

$$a = 2/9 = 0.2222,\ b = -2,\ c = 144,\ d_1 = 0.4,\ d_2 = 80.$$

   (Above are the true values.) [8]

   (ii) An accurate computer plot is shown on the next page. [6]

   (iii) The phase of $GK(j\omega)$ equals $-135°$ at the frequency 4 rad/sec (accurate value 4.27 rad/sec). The gain at this frequency is -38dB = 0.0126 approx (accurate value -39.03 dB = 0.0112). Hence the value of k is 79.4 (accurate value 89.45). [5]

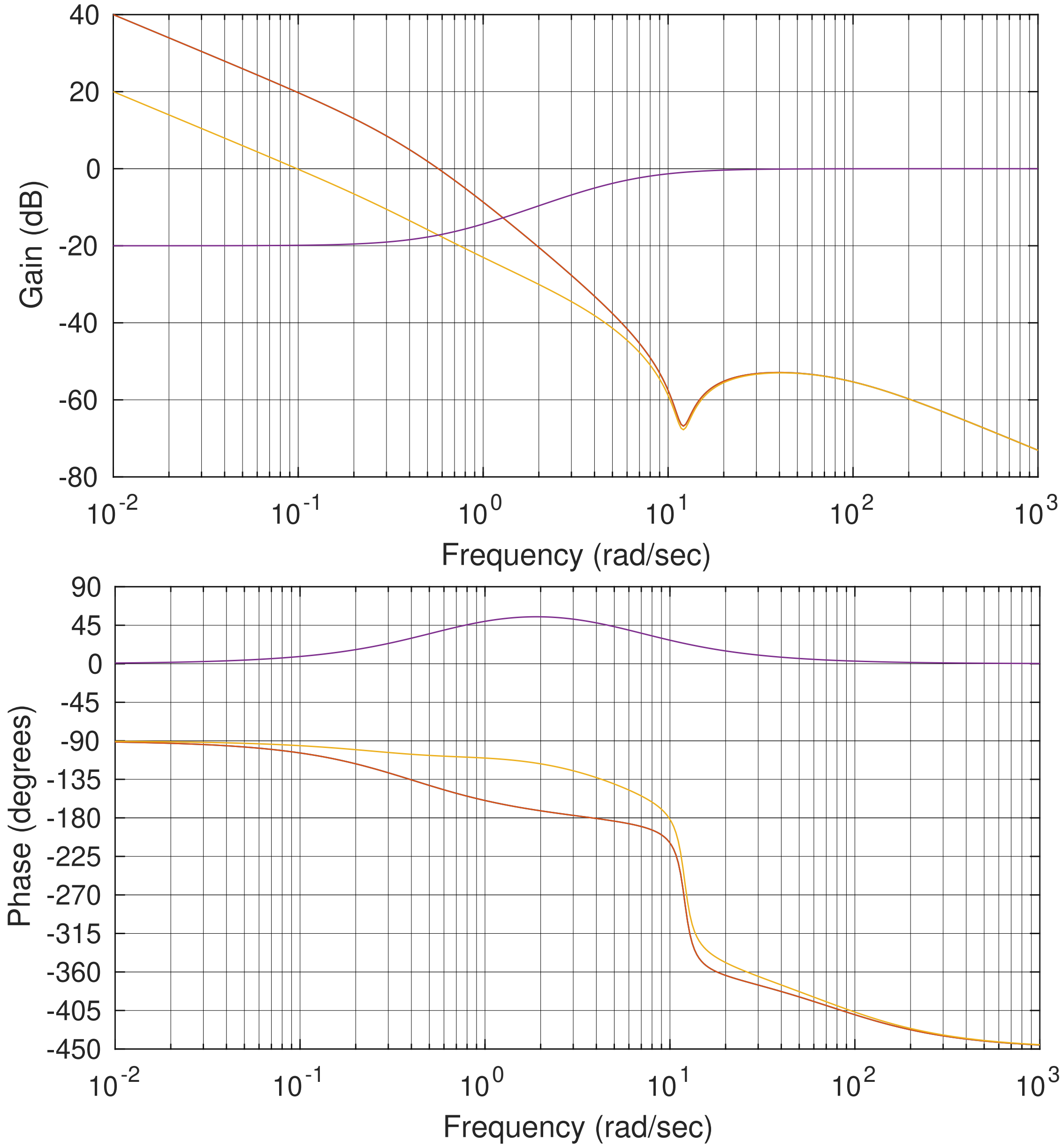
2. (a) Taking Laplace transforms gives the equations:

$$\begin{aligned}
\bar{p} &= k_0(\bar{r} - \bar{c}), \\
\bar{q} &= k_1 \frac{1}{s} \bar{p}, \\
\bar{c} &= \frac{1}{s}(k_2 \bar{p} + k_3 \bar{q} - \bar{v}) \\
&= \frac{1}{s}(k_2 \bar{p} + k_1 k_3 \frac{1}{s} \bar{p} - \bar{v}) \\
&= \frac{1}{s}((k_2 + k_1 k_3 \frac{1}{s}) k_0 (\bar{r} - \bar{c}) - \bar{v}).
\end{aligned}$$

   Hence the model can be expressed in the form of the block diagram with

$$\begin{aligned}
A &= k_0 k_2, \\
B &= k_0 k_1 k_3.
\end{aligned}$$

[6]



40
20
0
-20
-40
-60
-80
Gain (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)
90
45
0
-45
-90
-135
-180
-225
-270
-315
-360
-405
-450
Phase (degrees)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Figure 1:

(b) From the block diagram, or directly from the equations,

$$\begin{aligned} H_1(s) &= T_{r\to c} = \frac{\frac{As+B}{s^2}}{1+\frac{As+B}{s^2}} = \frac{As+B}{s^2+As+B}, \\ H_2(s) &= T_{v\to c} = \frac{\frac{-1}{s}}{1+\frac{As+B}{s^2}} = \frac{-s}{s^2+As+B}. \end{aligned}$$

The system is stable since the denominator is second order with all coefficients positive hence has all its roots are in the left half plane. [6]

(c) (i) When $A > 0$ and $B > 0$, $H_1(0) = 1$ and $H_2(0) = 0$ independent of variations in A and B. Hence there is accurate steady-state tracking of the reference value for calcium plasma concentration and no steady-state effect from changes in the steady-state calcium clearance rate, i.e. the model exhibits the two stated properties of the real biological system. [4]

(ii) When $A > 0$ and $B = 0$, $H_1(0) = 1$ independent of variations in A, but $H_2(0) = -A$. Hence the model still maintains accurate steady-state tracking of the reference value for calcium plasma concentration, however there will now be a steady-state effect from changes in the steady-state calcium clearance rate which will only be mitigated by large values of A. [4]

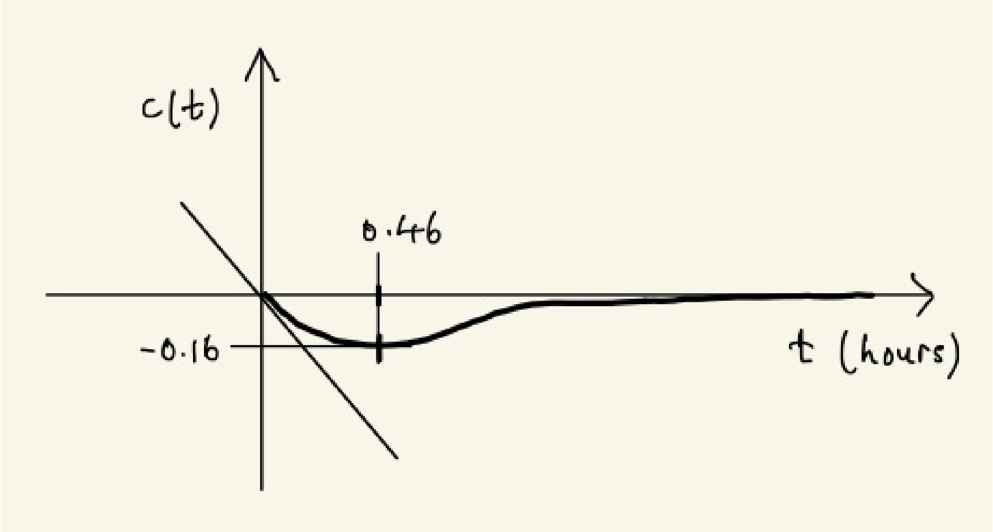
(d) With $A = 5$ and $B = 4$

$$H_2(s) = \frac{-s}{s^2+5s+4}$$

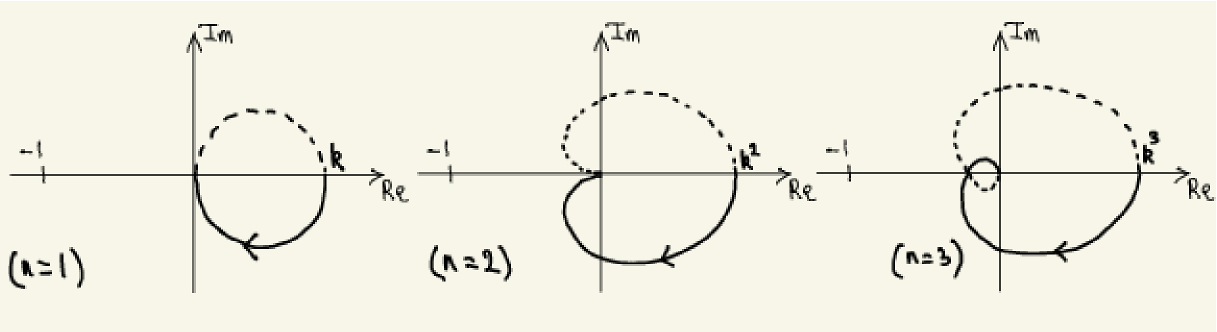
so the Laplace transform of the response $c(t)$ to a unit step in $v(t)$ equals

$$\frac{-1}{s^2+5s+4} = \frac{-1/3}{s+1} + \frac{1/3}{s+4}$$

and hence $c(t) = (-e^{-t} + e^{-4t})/3$. Then $\dot{c}(t) = (e^{-t} - 4e^{-4t})/3$ so initial slope is -1, minimum occurs at $t = \ln(4)/3 = 0.4621$ with value of -0.1575. Steady-state value is zero as expected. Sketch: [5]



3. (a)



For $n = 1$ or 2 the Nyquist diagram stays to the right of the -1 point for any $k > 0$ but not necessarily for $n = 3$ if k is large enough. [7]

(b) First crossing of negative real axis occurs when $n \arctan(\omega_0 T) = \pi$ which gives the required condition. At this frequency

$$|G(j\omega_0)| = \frac{k^n}{(T^2\omega_0^2 + 1)^{n/2}}.$$

Hence

$$\begin{aligned} |G(j\omega_0)| < 1 \quad &\Leftrightarrow \quad k^2 < T^2\omega_0^2 + 1 \\ &\Leftrightarrow \quad k^2 < \tan^2\left(\frac{\pi}{n}\right) + 1 = \sec^2\left(\frac{\pi}{n}\right) \end{aligned}$$

which is the condition to ensure that the Nyquist diagram stays to the right of the -1 point (which is the condition for stability of the feedback amplifier) and the required inequality follows. [6]

(c) From part (3b), $|G(j\omega)| = 1 \Leftrightarrow k^2 = T^2\omega^2 + 1$ from which the result follows. [3]

(d) (i) From part (3c), $|G(j\omega_1)| = 1$ when $\omega_1 T = \sqrt{k^2 - 1} = \sqrt{2}$. Hence the PM $= 180 - 3\tan^{-1}(\sqrt{2}) = 15.79°$. [4]

(ii) For the frequency $\omega = 30$ rad/sec, $\omega T = 30/\sqrt{300} = \sqrt{3}$. Hence $\angle G(j\omega) = -3\tan^{-1}(\sqrt{3}) = -180°$.

$$|G(j30)| = \frac{\sqrt{3}^3}{(3+1)^{3/2}} = 3\sqrt{3}/8 = 0.6495$$

i.e. $G(j30) = -0.6495$. The closed-loop transfer function $T(s) = G(s)/(1 + G(s))$, hence $T(j30) = -1.8532$. Hence the steady state response of the feedback amplifier is $-1.8532\sin(30t)$. [5]

SECTION B

4. (a) This is bookwork, as follows:

$$\int_{-\infty}^{\infty} x(t)^2 dt = \int_{-\infty}^{\infty} x(t)(\frac{1}{2\pi}\int X(\omega)\exp(j\omega t)d\omega)dt$$
$$= \int_{-\infty}^{\infty} X(\omega)(\int_{-\infty}^{\infty} x(t)\exp(j\omega t)dt)d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

[5]

(b) Use result for differentiation under Fourier transform (multiply transform by $j\omega$, see Info. Data book):

$$aj\omega Y(\omega) + Y(\omega) = X(\omega)$$

so

$$H(\omega) = Y(\omega)/X(\omega) = 1/(1 + ja\omega)$$

Not possible to increase energy since $|H(\omega)|^2 \leq 1$ and hence by Parseval in part a) the integrand is always non-zero and smaller than for x. [5]

(c) From definition of FT:

$$X(\omega) = \int_0^{\infty} \exp(-t)\exp(-j\omega t)dt = \frac{-1}{j\omega + 1}[\exp(-(t(1 + j\omega))]_0^{\infty} = \frac{1}{j\omega + 1}$$

since upper limit $\exp(-(t(1 + j\omega)) \overset{t\to\infty}{\to} 0$ for any finite ω.
Its energy is obtained from Parseval, as proven in part a):

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + 1}d\omega$$
$$= \frac{1}{2\pi}[\tan^{-1}(\omega)]_{-\infty}^{\infty} = \pi/(2\pi) = 0.5$$

using Maths databook for integral and noting $\tan^{-1}(\pm\infty) = \pm\pi/2$. [5]

(d) Use frequency response result from part b) to give $Y(\omega) = H(\omega)X(\omega)$ and hence

$$|Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2 = \frac{1}{|(1 + ja\omega)|^2}\frac{1}{|(1 + j\omega)|^2} = \frac{1}{(1 + a^2\omega^2)^2}\frac{1}{(1 + \omega^2)}$$
$$= \frac{1}{1 - a^2}\left(\frac{1}{(1 + \omega^2)} - \frac{a^2}{(1 + a^2\omega^2)}\right)$$

where we have split into partial fractions for use in working next (note the easiest way to do this PF in terms of ω^2 directly. Other valid solutions involve 4 PF terms and complex denominators, but will be more error-prone and slower.)

And so applying Parseval to y:

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1-a^2}\left(\frac{1}{(1+\omega^2)} - \frac{a^2}{(1+a^2\omega^2)}\right)d\omega$$
$$= \frac{1}{2\pi(1-a^2)}\left([\tan^{-1}(\omega) - a\tan^{-1}(\omega/a)]_{-\infty}^{\infty}\right)$$
$$= \frac{1}{2\pi(1-a^2)}(1-a)\pi = \frac{1}{2(1+a)}$$

Hence ratio of output to input energy is $E_y/E_x = 1/(2(1+a)) = 0.5$. Solving for a gives $a = 1$. [10]

5. (a) (i) Information Databook plus duality gives the Fourier transform as:

$$2\pi\Lambda(\omega/2)$$

where

$$\Lambda(t) = \max(0, 1-|t|)$$

is the triangle pulse of width [-1,1]. Hence largest non-zero component is just below $\omega = 2$, i.e. $\omega_{max} = 2$. [3]

(ii) With $T = 2\pi/3$ we have $\omega_0 = 2\pi/T = 3$. This is below the Nyquist frequency of $2\omega_{max}$. Spectrum is:

$$X_s(\omega) = \frac{1}{T}\sum_n X(\omega - n\omega_0) = 3\sum_n \Lambda((\omega - 3n)/2)$$

which has overlapping components and hence not reconstructable. Sketch: [4]

(iii) Nyquist frequency is 4, hence set $T = 2\pi/4 = \pi/2$ as suggested for perfect reconstruction. To reconstruct apply ideal reconstruction filter with frequency response:

$$H_r(\omega) = T\mathbf{1}(\omega \in [-2, +2])$$

with impulse response (from tables and duality):

$$h_r(t) = 2\, sinc(2t)$$

to the sampled signal:

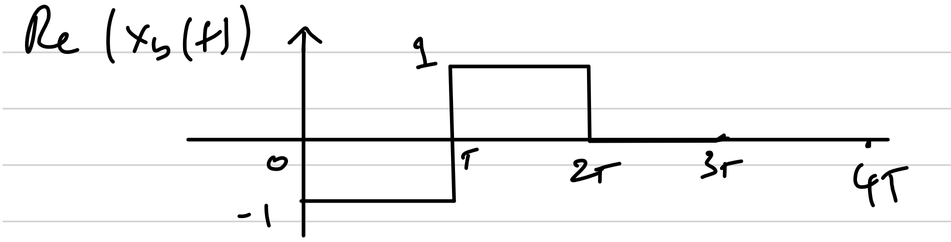
$$x_s(t) = \sum_n \delta(\tau - nT)x_n$$

Hence by convolution and use of sifting property of delta:

$$x(t) = \int h_r(t-\tau)x_s(\tau)d\tau = \int 2\, sinc((t-\tau)2)\sum_n \delta(\tau - nT)x_n d\tau$$
$$= 2\sum_n sinc((t - n\pi/2)2)x_n$$

[6]

(b) (i) The real parts of the symbols modulating $\cos(2\pi f_c t)$ are $(-1, 1, 0, 0)$. With a rectangular pulse of duration $T = 1$, the baseband waveform modulating $\cos(2\pi f_c t)$ is shown below. [5]



(ii) Since each symbol of the complex constellation has magnitude 1, the average energy is 1. The average energy of the real-valued constellation is

$$E_s = \frac{1}{4}\left[(-3A)^2 + (-A)^2 + (A)^2 + (3A)^2\right] = 5A^2$$

Setting $E_s = 1$, we have $A = \sqrt{1/5} = 0.447$. [3]

iii) For the complex-valued constellation, each symbol is $\sqrt{2}$ away from its two closest symbols. For the real-valued constellation, the distance between neighbouring symbols is $2A = 0.894$, which is smaller than $\sqrt{2}$. We therefore expect the error probability of the real-valued constellation to be larger, as the separation between neighbouring symbols is smaller. [4]

6. (a) We are given $x(t) = 3\sin(2000\pi t) + 4\cos(2000\pi t)$.

(i) The amplitude of $x(t) = \sqrt{3^2 + 4^2} = 5$, since $x(t)$ can be written as $\sqrt{5}\sin(2000\pi t + \phi)$, where $\phi = \tan^{-1}(4/3)$. [2]

ii) Assuming that the quantization noise n_Q is uniformly distributed between $[-\Delta/2, \Delta/2]$, as derived in the lecture notes, the power of the quantization noise is $\mathbb{E}[n_Q^2] = \Delta^2/12 = (1/3) \times 10^{-2} = 0.003333$. [4]

iii) Since the signal is a sinusoid with amplitude 5, the signal power $P = 5^2/2 = 12.5$. Therefore the SNR $= \frac{(5^2/2)}{(1/300)} = 3750 = 35.74$ dB. [3]

iv) To cover the whole range from -5 to +5 with uniformly spacing between levels $\Delta = 0.2$, the minimum number of levels required is $(10/\Delta) = 50$ (may need 51 levels to cover the whole range). [3]

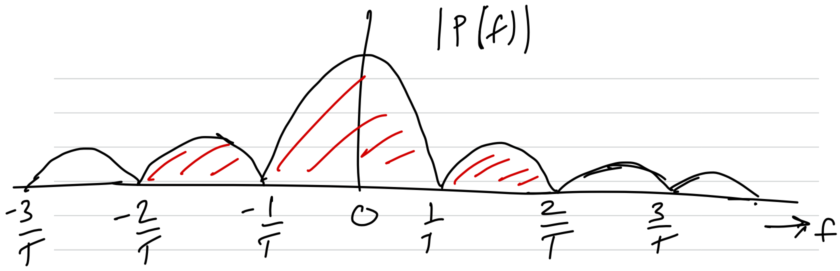
The minimum number of bits required is then given by $\lceil \log_2 51 \rceil = 6$ bits.

v) The signal is periodic with frequency $1000Hz$, so the sampling rate is $\frac{1}{T} = 1.2 \times 2 \times 1000 = 2400$ samples/sec. [4]

With a quantiser using 6 bits/samples, the bit rate is $14,400$ bits/second (14.4 kbps).

(b) i) With a four-symbol constellation, each symbol carries 2 bits, and the bit rate is $R = \frac{2}{T} = 500 \times 10^3$. This gives $T = 4 \times 10^{-6}$ s. [2]

ii) We are given that $p(t) = \frac{1}{\sqrt{T}}$ for $0 \le t < T$ and 0 otherwise. Therefore, from the data book: $|P(f)| = \sqrt{T}\text{sinc}(\pi f T)$. This is sketched below. (Note that shifting the pulse by a constant amount t_0 does not change the magnitude of $P(f)$.) [4]



iii) The shaded area in the figure is the part of the spectrum that determines the band-pass bandwidth of each user. Therefore, the effective band-pass bandwidth per user is $4/T = 10^6$ Hz = 1 MHz. Since the total bandwidth is 100 MHz, the number of users that can be accommodated is 100. [3]

M.C. Smith, S.J. Godsill, R. Venkataramanan
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