EGT1
ENGINEERING TRIPOS PART IB

Thursday 12 June 2025   14.00 to 16.10

Paper 6

INFORMATION ENGINEERING

Answer not more than ***four*** *questions.*

Answer not more than **two** *questions from each section.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Supplementary pages: one extra copy of Figure 1 for Question 1
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

Answer not more than **two** *questions from this section*

1 (a) (i) What does it mean for a Linear Time-Invariant (LTI) system to be asymptotically stable? [2]

(ii) Suppose a LTI system can be realised in a physical system that can be described by ordinary differential equations. What conditions should its transfer function satisfy? [2]

(b) Figure 1 shows the Bode diagram of a linear system with transfer function

$$G(s) = \frac{a(1 + 0.2T_1 s + T_1^2 s^2)(1 + T_3 s)}{(1 + 0.5T_2 s + T_2^2 s^2)(1 + T_4 s)(1 + 100s)}.$$

Estimate the values of a, T_1, T_2, T_3 and T_4. [8]

(c) Suppose the system $G(s)$ in part (b) is controlled in a unity gain negative feedback system with a controller $K(s)$, as shown in Figure 2.

(i) Suppose $K(s) = k_1$. What is the gain margin of the system with this controller? [2]

(ii) Find a value of k_1 for which the phase margin is 80°. [2]

(iii) Suppose now that $K(s) = k_2 \frac{(1+s)}{(10+s)}$. Assuming a value of $k_2 = 10$, sketch the Bode diagram of the controller $K(j\omega)$ and the return ratio $G(j\omega)K(j\omega)$ on the attached copy of Figure 1. [5]

(iv) Estimate the phase margin with the choice of controller in part (iii). [4]

An additional copy of Figure 1 is attached to the back of this paper. This should be detached and handed in with your answers.

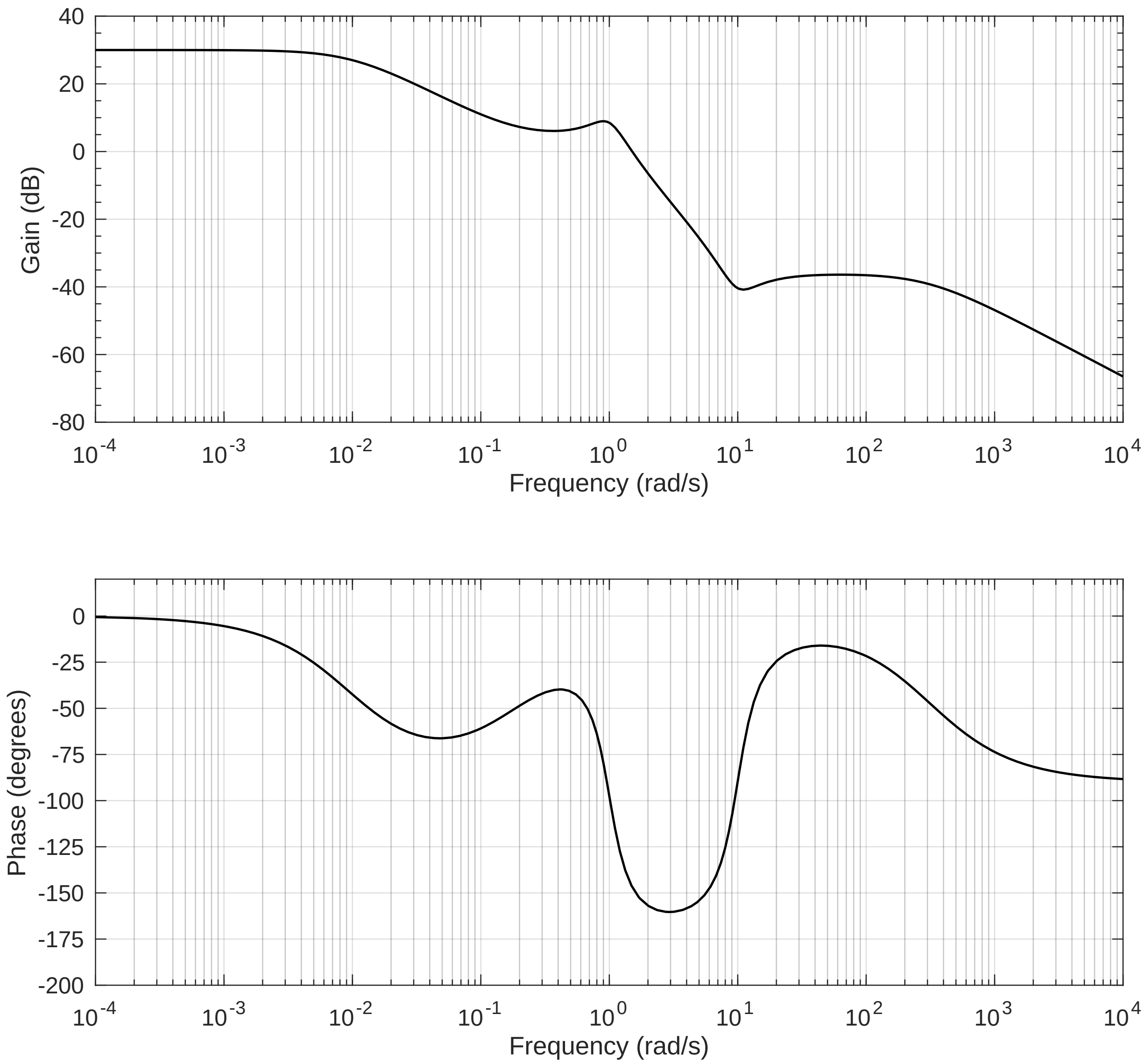

Figure 1

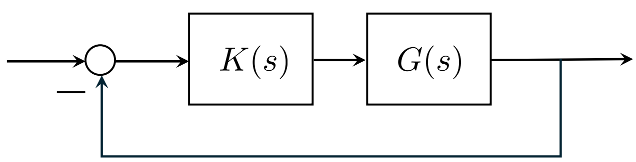

Figure 2

2 (a) Figure 3 shows the Nyquist diagrams for positive frequencies of six systems, $G_1, \ldots G_6$, with the following transfer functions:

$$G_1 = \frac{1+2s}{1+s};\ G_2 = \frac{s+2}{1-2s};\ G_3 = \frac{1+2s}{s(1+s/10)};$$

$$G_4 = \frac{2}{s(1+0.3s)(1+0.1s)};\ G_5 = \frac{2}{(1+0.3s)(1+0.3s)};\ G_6 = \frac{\exp(-s/100)}{(1+s)(1+5s)}.$$

Using clear reasoning, assign each of the systems to the label (A-F) of its corresponding diagram, giving your answer in the form, "$G_x \to (Y)$ because ....". [8]

(b) The transfer function of a plant, G, is given by

$$G(s) = \frac{1}{s(1+T_1 s)(1+T_2 s)},\ \text{with } T_1, T_2 > 0.$$

(i) Evaluate the asymptotic magnitude and phase of the frequency response $G(j\omega)$ as $\omega \to 0$ and for $\omega \to \infty$, with $\omega > 0$. [5]

(ii) Does the Nyquist diagram of $G(j\omega)$ cross the real axis at any finite, nonzero frequency? Depending on your answer, either find expression(s) for any such crossing, or prove that it does not cross the real axis. [4]

(iii) Sketch the Nyquist diagram of G. [3]

(iv) G is controlled using proprtional negative feedback with gain K. For what values of K is the closed loop system asymptotically stable? [5]

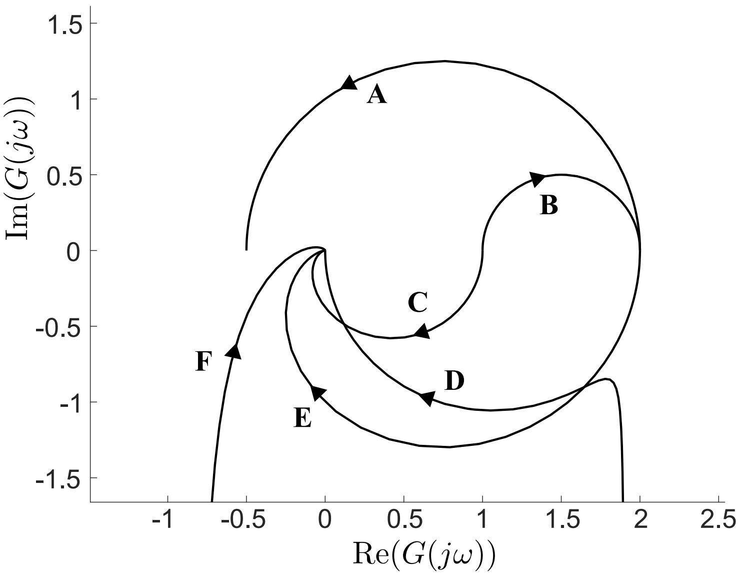

Figure 3

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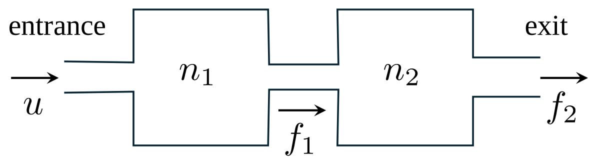

Figure 4

3 Figure 4 shows the plan of an exhibition venue consisting of two adjoining halls that contain n_1 and n_2 visitors. Visitors enter the venue at a rate u into the first hall, move at a rate f_1 between the halls, and leave the venue from the second hall at a rate f_2. The visitor flow rates are modelled as continuous variables by the following equations, with constants $\alpha, \beta > 0$:

$$\dot{n}_1 = u - f_1; \qquad f_1 = \alpha(n_1 - n_2);$$
$$\dot{n}_2 = f_1 - f_2; \qquad f_2 = \beta n_2.$$

(a) (i) Compute the transfer function $G_1(s)$ from u to f_1 [4]

(ii) Compute the transfer function $G_2(s)$ from f_1 to f_2 [3]

(iii) Using your answers to parts (i) and (ii), or otherwise, show that the transfer function $G(s)$ from u to f_2 has the form

$$G(s) = \frac{P}{s^2 + Qs + R},$$

and find expressions for P, Q and R. [3]

(b) The empty venue receives a step increase in visitor rate, u, from 0 to a constant, A, at time $T > 0$.

(i) Sketch the exit rate $f_2(t)$ as a function of time. [3]

(ii) Determine whether the exit rate can at any time exceed the entrance rate, for any physically realisable entrance rate function, $u(t)$. [5]

(c) The venue management introduce a proportional negative feedback control with gain k from the exit rate to the entrance rate, in order to track a reference entrance rate, $u_{\text{ref}}(t)$.

(i) Draw a block diagram of the system. [3]

(ii) Write down the new (closed loop) transfer function from entrance to exit rates, and calculate the steady-state outflow rate given a reference rate that steps from a steady value of 0 to A at time $t = 0$. [4]

SECTION B

Answer not more than **two** *questions from this section*

4 (a) Consider a continuous-time signal defined as:

$$x(t) = \text{sgn}(t)e^{-a|t|}, \quad a > 0$$

where:

$$\text{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0 \end{cases}$$

(i) Show that the Fourier Transform of $x(t)$ is:

$$X(\omega) = \frac{-2j\omega}{a^2 + \omega^2}$$

[3]

(ii) Sketch its magnitude and phase spectra. [3]

(b) Let $f(t) = e^{-a|t|}, \quad a > 0$.
Verify the following relation by separately evaluating the left and right hand sides:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

HINT:

$$\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^2} dx = \frac{\pi}{2}$$

[11]

(c) Determine the Fourier transform of the following signals $x(t), h(t)$, and compute the convolution $x(t) * h(t)$ using the Convolution Theorem:

$$x(t) = \text{sgn}(t)e^{-a|t|}, \quad h(t) = u(t) - u(t - T)$$

where $u(t)$ denotes the unit step function. [8]

5 (a) Consider a signal given by:

$$x(t) = \cos(2000\pi t),$$

where t is in seconds. This signal is sampled at a frequency of $f_s = 1500\,\text{Hz}$.

(i) Determine the frequency of the original signal in Hz. [1]

(ii) Calculate the Nyquist frequency for this sampling rate. [2]

(iii) Compare the original signal's frequency to the Nyquist frequency. If these frequencies differ, calculate the frequency (in Hz) of the signal that will actually be observed if we attempt to reconstruct the signal from samples using an ideal reconstruction filter with cutoff frequency 750 Hz. Explain your reasoning. [4]

(b) Calculate the Discrete Fourier Transform (DFT) of a discrete-time signal:

$$x[n] = \{1, 2, 0, -1\}, \quad N = 4.$$

[5]

(c) A message signal $m(t) = 2\cos(1000\pi t) - \cos(2000\pi t)$ volts is transmitted using frequency modulation (FM) with a carrier frequency of f_c. The transmitted signal is

$$s(t) = \cos\left(2\pi f_c t + 2\pi k_\text{f} \int_0^t m(u)du\right).$$

Assume that $f_c = 1$ MHz and $k_\text{f} = 3000$ Hz V^{-1}.

(i) Determine the spectrum $M(f)$ of the message signal. [3]

(ii) Compute the modulation factor of the FM signal $s(t)$ and estimate its bandwidth. [5]

(iii) Briefly describe how you can recover $m(t)$ from the FM signal $s(t)$. You can ignore the effect of noise. [5]

6 A digital passband transmitter transmits a waveform

$$x(t) = x_b(t)\cos(2\pi f_c t),$$

where $x_b(t)$ is a pulse amplitude modulated signal given by $x_b(t) = \sum_k X_k p(t - kT)$, where the pulse $p(t) = \frac{1}{\sqrt{T}}\text{sinc}\left(\frac{\pi t}{T}\right)$. Here the information symbols $X_0, X_1, X_2 \ldots$ are drawn from the constellation $\{0, A\}$, $T = 10^{-4}$s, and $f_c = 10$MHz. The received waveform is $y(t) = x(t) + n(t)$, where $n(t)$ is a noise waveform.

(a) What is the rate of transmission, in bits per second? [2]

(b) Compute the spectrum $P(f)$ of the pulse, and hence determine the passband bandwidth of the transmitted waveform $x(t)$. [5]

(c) We use a two-stage receiver to recover the information symbols from $y(t)$. The first stage involves a product modulator followed by a low-pass filter to produce $y_b(t)$. In the second stage, $y_b(t)$ is passed through a matched filter whose output is sampled at times mT, for $m = 0, 1, 2, \ldots$. The sampled matched filter output is $Y_m = X_m + N_m$, where N_m is independent Gaussian noise with mean zero and variance σ^2.

(i) Draw a block diagram of the two-stage receiver, indicating the key details, including the waveform used in the product modulator and the impulse response of the matched filter. [5]

(ii) For the optimum decision rule, compute the probability of detection error. Express your answer in terms of the ratio $\frac{A^2}{\sigma^2}$ and the Q-function, where $Q(u) = 1 - \Phi(u)$, and $\Phi(u)$ is the Gaussian cumulative distribution function. You may assume that all the constellation symbols are equally likely. [4]

(iii) Express the result of part (ii) in terms of $\frac{E_b}{\sigma^2}$, where E_b is the average energy per bit of the constellation. Using the approximation $Q(u) \approx \frac{1}{2}e^{-u^2/2}$, estimate the minimum signal-to-noise ratio $\frac{E_b}{\sigma^2}$ (in dB) required to obtain a probability of detection error of 0.01. [4]

(iv) Now assume that the underlying information bits are first encoded using a $(7, 4)$ Hamming code. Each *coded* bit is mapped to a symbol in $\{0, A\}$ to generate the transmitted waveform $x(t)$, as above. At the receiver, after demodulation and detection, the Hamming decoder is applied to recover the information bits.
Assuming that $\frac{E_b}{\sigma^2}$ is set to the same value as in part (c)(iii), what is the probability of an *information* bit being decoded in error? What is the effective transmission rate of this coded system? [5]

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Version TO/5 Candidate Number:

EGT1
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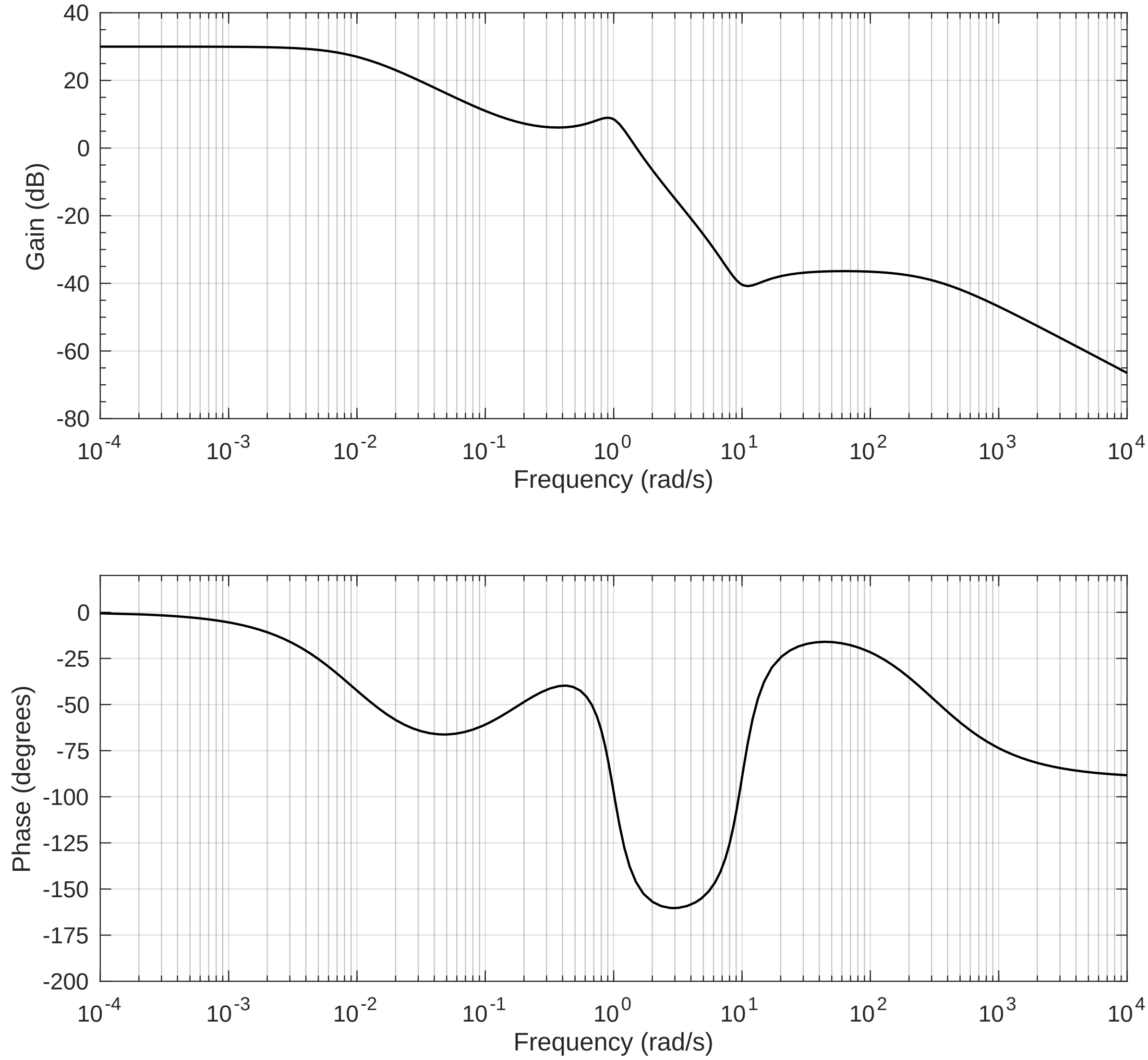
Thursday 12 June 2025, Paper 6, Question 1.



Extra copy of Figure 1 for Question 1.

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