Solutions to

Ib P7

Q1 - Q3

E. Mastorakos

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Q1(a) $f(x,y) = e^{-(x^2+y^2)}$, $g(x,y) = \frac{y}{\sqrt{x^2+y^2}}$

Let $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow f(x,y) = e^{-r^2}$, $g(x,y) = \sin\theta$

$$I = \iint fg\,dx\,dy = \int_0^R \int_0^{\pi/4} e^{-r^2} \sin\theta \; J \; dr\,d\theta$$

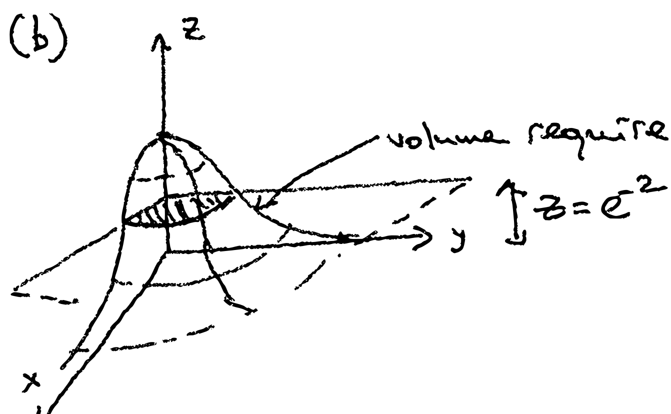
(the limit $\pi/4$ is because we are limited by the $x=y$ line, i.e. $\theta = \pi/4$)

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = r$$

$$\therefore I = \int_0^R \int_0^{\pi/4} e^{-r^2} \sin\theta \; r\,dr\,d\theta = \int_0^R r e^{-r^2} dr \int_0^{\pi/4} \sin\theta\,d\theta$$

$$= \left[-\frac{1}{2}e^{-r^2}\right]_0^R \left[-\cos\theta\right]_0^{\pi/4} = \frac{1}{2}\left(1-e^{-R^2}\right)\left(1-\frac{1}{\sqrt{2}}\right).$$

(b)



Q1 (c)

Volume under surface can be found by repeating Part (a) with $g=1$ $\Rightarrow$

whole volume under curve up to radius R $= \int_0^{2\pi}\int_0^R e^{-r^2} r\,dr\,d\theta$

$= (1-e^{-R^2})\pi$

Volume under whole surface is above expression for $R\to\infty$ ($z=0$), i.e. π.

Volume between $z=0$ & $z=e^{-2}$ (i.e. $R=\sqrt{2}$)

is $\pi - (1-e^{-2})\pi = \pi/e^2$

[Alternatively: required volume is

$$\int_{\sqrt{2}}^{\infty}\int_0^{2\pi} e^{-r^2} r\,dr\,d\theta = \int_{\sqrt{2}}^{\infty} re^{-r^2}dr \int_0^{2\pi} d\theta = \left[-\tfrac{1}{2}e^{-r^2}\right]_{\sqrt{2}}^{\infty}\left[2\pi\right]$$

$= \pi/e^2$.]

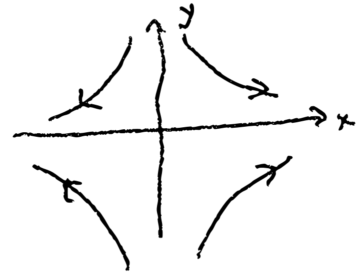
Now we must add the volume of the cylinder of height $z=e^{-2}$ and radius $\sqrt{2}$

i.e. $\pi\cdot 2\cdot e^{-2}$ $\Rightarrow$ required volume $= \underline{\underline{3\pi e^{-2}}}$.

Q2 (a) Field lines obey $\frac{dy}{dx} = \frac{u_y}{u_x} \Leftrightarrow \frac{dy}{dx} = -\frac{Ay}{Ax}$

$$\Rightarrow \frac{dy}{y} = -\frac{dx}{x} \Leftrightarrow \ln y = -\ln x + c'$$

$$\Leftrightarrow y = \frac{c}{x}$$



(b) Solenoidal if $\nabla \cdot \underline{u} = 0$

$$\nabla \cdot \underline{u} = \frac{\partial(Ax)}{\partial x} + \frac{\partial(-Ay)}{\partial y} = A - A = 0 \qquad \text{QED}$$

Irrotational if $\nabla \times \underline{u} = 0$

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax & -Ay & 0 \end{vmatrix} = 0 \qquad \text{QED}$$

(c) Since the field is irrotational, it has a scalar potential Φ such that $\underline{u} = \nabla\Phi$. Hence,

$$\frac{\partial \Phi}{\partial x}\underline{i} + \frac{\partial \Phi}{\partial y}\underline{j} = (Ax)\underline{i} - (Ay)\underline{j}$$

For x : $\frac{\partial \Phi}{\partial x} = Ax \Leftrightarrow \Phi = \frac{Ax^2}{2} + f_x(x)$

For y : $\frac{\partial \Phi}{\partial y} = -Ay \Leftrightarrow \Phi = -\frac{Ay^2}{2} + f_y(y)$

$$\Rightarrow \Phi = \frac{A}{2}(x^2 - y^2) + c$$

Q2 (c) cont'd

Required line integral $\left(\int 2\underline{u}\cdot d\underline{l} = 2\int \underline{u}\cdot d\underline{l}\right.$

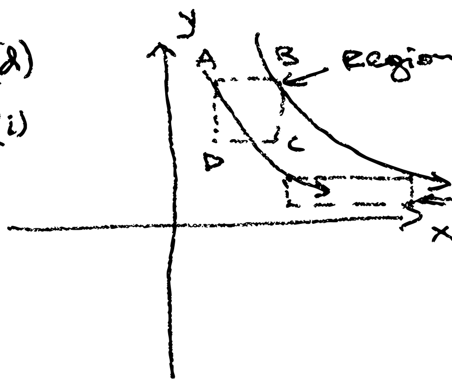
$= 2 \times$ (scalar difference between the two points)

$\Rightarrow \int 2\underline{u}\cdot d\underline{l} = 2\left[\Phi(0,1) - \Phi(1,0)\right] = -2A$

NOTE: $2A$ is also OK since direction is not explicitly given

(d)

(i)



Since $\dfrac{dX}{dt} = u_x$ & $\dfrac{dY}{dt} = u_y \Rightarrow \dfrac{dX}{dt} = AX,\ \dfrac{dY}{dt} = -AY$

$\Rightarrow X = X_0 e^{At}$ & $Y = Y_0 e^{-At}$ with X_0, Y_0 the initial point at $t=0$.

Therefore the distance $AB = (X_0+L)e^{At} - X_0 e^{At}$

$= Le^{At}$ QED

Similarly, $BC = Le^{-At}$

(ii) $\Rightarrow$ Area $= AB\cdot BC = L^2 =$ initial area

Alternatively, from the Divergence theorem,

since $\nabla\cdot\underline{u} = 0 \Rightarrow \int_S \underline{u}\cdot\underline{n}\,ds = 0 \Rightarrow$ area unchanged

Q3 (a)

$$x = \eta\sqrt{4\alpha t} \quad \Rightarrow \quad \eta = \frac{x}{\sqrt{4\alpha}\sqrt{t}}$$

$$\frac{\partial\eta}{\partial t} = \frac{x}{\sqrt{4\alpha}}\left(-\frac{1}{2}\right)t^{-3/2} = \left(-\frac{1}{2}\right)\eta\, t^{-1} = -\frac{\eta}{2t}$$

$$\frac{\partial\eta}{\partial x} = \frac{1}{\sqrt{4\alpha}\sqrt{t}}$$

$$g = \frac{M}{\sqrt{4\alpha t}} = A t^{-1/2}$$
(for convenience)

(b)
$$\frac{\partial c}{\partial t} = \frac{dg}{dt}f + g\frac{df}{dt}$$

$$= -\frac{1}{2}At^{-3/2}f + At^{-1/2}\left(-\frac{1}{2}\right)\eta t^{-1}\frac{df}{d\eta}$$

$$\Rightarrow \frac{\partial c}{\partial t} = -\frac{1}{2}At^{-3/2}\left(f + \eta\frac{df}{d\eta}\right) \qquad (1)$$

$$\frac{\partial^2 c}{\partial x^2} = g\frac{d^2 f}{dx^2} = g\frac{d}{dx}\left(\frac{df}{dx}\right) = g\frac{d}{d\eta}\left(\frac{df}{dx}\right)\frac{\partial\eta}{\partial x}$$

$$= g\frac{\partial}{\partial\eta}\left(\frac{\partial f}{\partial\eta}\frac{\partial\eta}{\partial x}\right)\frac{\partial\eta}{\partial x}$$

$$= g\frac{\partial^2 f}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + \frac{\partial f}{\partial\eta}\cancelto{0}{\frac{\partial}{\partial\eta}\left(\frac{\partial\eta}{\partial x}\right)}\frac{\partial\eta}{\partial x}$$

$$= g\cdot\frac{d^2 f}{d\eta^2}\left(\frac{1}{\sqrt{4\alpha}}\right)^2 t^{-1} \qquad (2)$$

Therefore $\dfrac{\partial c}{\partial t} = \alpha \dfrac{\partial^2 c}{\partial x^2}$ becomes (using ① & ② from above):

$$\left(-\frac{1}{2} A t^{-3/2}\right)\left(f + \eta \frac{df}{d\eta}\right) = \alpha \frac{1}{4\alpha} t^{-1} \frac{d^2 f}{d\eta^2} \cdot A t^{-1/2}$$

$$\Rightarrow \frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} + 2f = 0 \qquad \text{QED}$$

(c) Starting from the ODE, one can write

$$\frac{d}{d\eta}\left(\frac{df}{d\eta} + 2\eta f\right) = 0 \Rightarrow \frac{df}{d\eta} + 2\eta f = C$$

Because of the b.c.'s, $f \to 0$ at $\eta \to \pm\infty$ upon integration of the above we must keep $C = 0$. This has the solution

$f = A_0 \exp(-\eta^2)$, A_0 to come from initial condition.

Note: the above sounds a little arbitrary & "magic", a rigorous derivation involves careful work around the singularity $t \to 0$. The above is acceptable.

Q3 (c) Cont'd

An alternative, equally acceptable, method is to start from the solution given, evaluate $\frac{\partial c}{\partial t}$ & $\alpha \frac{\partial^2 c}{\partial x^2}$ separately & show that they are equal. The differentiations are a little lengthy, the main steps are given below:

$$c = \frac{M}{\sqrt{4\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial t}\left(\frac{M}{\sqrt{4\alpha t}}\right) \exp\left(-\frac{x^2}{4\alpha t}\right) + \frac{M}{\sqrt{4\alpha t}} \frac{\partial}{\partial t}\left(\exp\left(-\frac{x^2}{4\alpha t}\right)\right)$$

$$= c\left(\frac{x^2}{4\alpha t^2} - \frac{1}{2t}\right) \qquad (1)$$

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right) = \frac{M}{\sqrt{4\alpha t}} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \exp\left(-\frac{x^2}{4\alpha t}\right)\right)$$

$$\Rightarrow \alpha\frac{\partial^2 c}{\partial x^2} = c\left(\frac{x^2}{4\alpha t^2} - \frac{1}{2t}\right) \qquad (2)$$

$$\Rightarrow (1) = (2) \qquad \text{QED}$$

Q4 (a)

$$\det(A) = \begin{vmatrix} 1 & 3 & k \\ k & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3 \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} k & 1 \\ 1 & 2 \end{vmatrix} =$$

$$= -1 - 3k + 3 + 2k^2 - k = 2(k-1)^2$$

IF $k \neq 1$ THEN SOLUTION IS UNIQUE FIND BY GAUSSIAN ELIMINATION

$$\begin{vmatrix} 1 & 3 & k \\ k & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 3 \\ 2 \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & 1 & k-1 \\ 0 & 1-2k & 1-k \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ 3-2k \\ 2 \end{vmatrix} \Rightarrow$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & k-1 \\ 0 & 0 & (1-k)-(k-1)(1-2k) \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ 3-2k-(-1)(1-2k) \\ 2 \end{vmatrix} \Rightarrow$$

$$\begin{vmatrix} 0 & 1 & k-1 \\ 0 & 0 & (1-k)(1+1-2k) \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} -1 \\ 4-4k \\ 2 \end{vmatrix}$$

$Z = \frac{2}{1-k}$ $Y = 1$ $X = -\frac{2}{1-k}$

IF $k = 1$ THEN $\begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$Y = -1$, $X = 4 - Z$, $Z \in \mathbb{R}$

Q4(b)(i)

$$A=\begin{pmatrix}1&3&2\\0&1&2\\-1&1&3\\0&0&\frac{1}{2}\end{pmatrix} \qquad \underline{q_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}} \quad \tilde{a}_2=\begin{bmatrix}3\\1\\1\\0\end{bmatrix}-\frac{1}{2}\left(\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\\0\end{bmatrix}\right)\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}=$$

$$=\begin{bmatrix}3\\1\\1\\0\end{bmatrix}-\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}=\begin{bmatrix}2\\1\\2\\0\end{bmatrix}$$

$$\underline{q_2=\frac{1}{3}\begin{bmatrix}2\\1\\2\\0\end{bmatrix}} \quad \tilde{a}_3=\begin{bmatrix}2\\2\\3\\\frac{1}{2}\end{bmatrix}-\frac{1}{2}\left(\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\2\\3\\\frac{1}{2}\end{bmatrix}\right)\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}-\frac{1}{9}\left(\begin{bmatrix}2\\1\\2\\0\end{bmatrix}\cdot\begin{bmatrix}2\\2\\3\\\frac{1}{2}\end{bmatrix}\right)\begin{bmatrix}2\\1\\2\\0\end{bmatrix}=$$

$$=\begin{bmatrix}2\\2\\3\\\frac{1}{2}\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}-\frac{12}{9}\begin{bmatrix}2\\1\\2\\0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}12+3-16\\12-0-8\\18-3-16\\3\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-1\\4\\-1\\3\end{bmatrix}$$

$$\underline{q_3=\frac{1}{3\sqrt{3}}\begin{bmatrix}-1\\4\\-1\\3\end{bmatrix}} \qquad Q=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{2}{3}&-\frac{1}{3\sqrt{3}}\\0&\frac{1}{3}&\frac{4}{3\sqrt{3}}\\-\frac{1}{\sqrt{2}}&\frac{2}{3}&-\frac{1}{3\sqrt{3}}\\0&0&\frac{3}{3\sqrt{3}}\end{bmatrix} \qquad A=QR \Rightarrow Q^TA=R$$

$$R=\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}&0\\\frac{2}{3}&\frac{1}{3}&\frac{2}{3}&0\\-\frac{1}{3\sqrt{3}}&\frac{4}{3\sqrt{3}}&-\frac{1}{3\sqrt{3}}&\frac{3}{3\sqrt{3}}\end{bmatrix}}_{Q^T}\underbrace{\begin{bmatrix}1&3&2\\0&1&2\\-1&1&3\\0&0&\frac{1}{2}\end{bmatrix}}_{A}=\begin{bmatrix}\frac{2}{\sqrt{2}}&\frac{2}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&3&4\\0&0&\frac{\sqrt{3}}{2}\end{bmatrix}$$

$$QR=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{2}{3}&-\frac{1}{3\sqrt{3}}\\0&\frac{1}{3}&\frac{4}{3\sqrt{3}}\\-\frac{1}{\sqrt{2}}&\frac{2}{3}&-\frac{1}{3\sqrt{3}}\\0&0&\frac{3}{3\sqrt{3}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{2}}&\frac{2}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&3&4\\0&0&\frac{\sqrt{3}}{2}\end{bmatrix}=\begin{bmatrix}1&3&2\\0&1&2\\-1&1&3\\0&0&\frac{1}{2}\end{bmatrix}=A$$

Q4 (b) (ii)

$A \in \mathbb{R}^{4\times 3}$

SINCE RANK(A) = 3 AND 4 ≥ 3 LEAST SQUARE SOLUTION SATISFIES: $A^T A x = A^T v$

$(QR)^T QR x = (QR)^T v \Rightarrow R^T Q^T Q R x = R^T Q^T v \Rightarrow$

$\Rightarrow R^T R x = R^T Q^T v$ (SINCE $Q^T Q = I$ AS Q - ORTHOGONAL)

$\Rightarrow Rx = Q^T v$ (SINCE R - NON-SINGULAR)

$$\begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 3 & 4 \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3\sqrt{3}} & \frac{4}{3\sqrt{3}} & -\frac{1}{3\sqrt{3}} & \frac{3}{3\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$e = \frac{2}{\sqrt{3}} \left(\frac{-1+8-1+6}{3\sqrt{3}} \right) = \frac{8}{3} \qquad c = \frac{+\frac{52}{9\sqrt{2}} + \frac{8}{3\sqrt{2}}}{\frac{2}{\sqrt{2}}} = \frac{38}{9}$$

$$d = \frac{\frac{2+2+2}{3} - \frac{32}{3}}{3} = -\frac{26}{9}$$

HENCE LEAST SQUARES SOLUTION IS $x = \begin{bmatrix} 4\frac{2}{9} \\ -2\frac{8}{9} \\ 2\frac{2}{3} \end{bmatrix}$

Q4 (c)

$$A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} a-\lambda & b & b \\ b & a-\lambda & b \\ b & b & a-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (a-\lambda)\begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} - b\begin{vmatrix} b & b \\ b & a-\lambda \end{vmatrix} + b\begin{vmatrix} b & a-\lambda \\ b & b \end{vmatrix}$$

$$= (a-\lambda)\left((a-\lambda)^2 - b^2\right) - b\left((a-\lambda)b - b^2\right) + b\left(b^2 - b(a-\lambda)\right)$$

$$= (a-\lambda)^3 - b^2(a-\lambda) - 2b^2(a-\lambda) + 2b^3 =$$

$$= (a-\lambda)^3 - 3(a-\lambda)b^2 + 2b^3 =$$

$$= (a-\lambda)^3 - 3(a-\lambda)b^2 + 3b^3 - b^3 =$$

$$= (a-\lambda-b)\left((a-\lambda)^2 + (a-\lambda)b + b^2\right) - 3b^2(a-\lambda-b)$$

$$= (a-\lambda-b)\left((a-\lambda)^2 + (a-\lambda)b - 2b^2\right) =$$

$$= (a-\lambda-b)\left((a-\lambda-b)(a-\lambda+b) + b(a-\lambda-b)\right) =$$

$$= (a-\lambda-b)^2(a-\lambda+2b)$$

SO EIGENVALUES: $a-b$ AN $a+2b$

$\text{DET}(A) = (a-b)^2(a+2b)$

EIGENVECTORS: FOR $a+2b \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

FOR $a-b \rightarrow$ ANY 2 VECTORS SPANNING $x+y+z=0$ E.G. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Q4(c)(ii)

EASY TO SHOW THAT $\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}$ IS AN EIGENVECTOR

WITH AN EIGENVALUE $a+(n-1)b$

SIMILARLY,

$$\begin{bmatrix}1\\\vdots\\1\\-(n-1)\end{bmatrix} \begin{bmatrix}1\\\vdots\\1\\n-1\\1\end{bmatrix} \cdots \begin{bmatrix}1\\n-1\\1\\\vdots\\1\end{bmatrix}$$ SPAN

SPACE OF $n\times 1$ VECTORS ORTHOGONAL TO $\begin{bmatrix}1\\1\\1\\\vdots\\1\end{bmatrix}$.

FOR EIGENVALUE $a-b$.

$DET(A) = (a-b)^{n-1}(a+(n-1)b)$ (PRODUCT OF EIGENVALUES)

Q4 (d)(i) · 6/6

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) \overset{(1)}{=} \det(A)$$

(1) SINCE $\det(P^{-1}) = \frac{1}{\det(P)}$

$$\text{RANK}(B) = \text{RANK}(P^{-1}AP) \overset{(2)}{=} \text{RANK}(AP) \overset{(2)}{=} \text{RANK}(A)$$

(2) SINCE $\text{RANK}(A) = \text{RANK}(UA) = \text{RANK}(AV)$ IF U, V — NON-SINGULAR

$$\text{DET}(B-\lambda I) = \text{DET}(P^{-1}AP - \lambda P^{-1}IP) = \text{DET}(P^{-1}(A-\lambda I)P) \overset{(1)}{=}$$
$$= \det(A-\lambda I)$$

HENCE EIGEN VALUES OF B AND A ARE SAME.

(d)(ii) PROVEN BY COUNTER-EXAMPLE

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ HAVE SAME DET, RANK, EIGEN VALUES

HOWEVER $P^{-1}IP = I \neq A$ FOR ALL INVERTIBLE MATRICES P.

Q5(a)(i)

$$\begin{bmatrix}1&2&1&3\\2&7&3&5\\-1&1&0&-4\end{bmatrix}-\begin{array}{c}1\\2\\-1\end{array}\begin{array}{c}1\;\;2\;\;1\;\;3\\ \begin{bmatrix}1&2&1&3\\2&4&2&6\\-1&-2&-1&-3\end{bmatrix}\end{array}=\begin{array}{c}0\\1\\1\end{array}\begin{array}{c}0\;\;3\;\;1\;\;-1\\ \begin{bmatrix}0&0&0&0\\0&3&1&-1\\0&3&1&-1\end{bmatrix}\end{array}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$L=\begin{bmatrix}1&0&0\\2&1&0\\-1&1&0\end{bmatrix}\quad U=\begin{bmatrix}1&2&1&3\\0&3&1&-1\\0&0&0&0\end{bmatrix}$$

(a)(ii) COL SPACE $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$, LEFT NULL SPACE $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}\times\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}3\\-1\\1\end{bmatrix}\right\}$ ROW SPACE $\left\{\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}0\\3\\1\\-1\end{bmatrix}\right\}$

NULL SPACE $\begin{bmatrix}1&2&1&3\\0&3&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow \begin{array}{l}x=-\frac{1}{3}\\y=-\frac{1}{3}\end{array}\Rightarrow$ $\left\{\begin{bmatrix}-\frac{1}{3}\\-\frac{1}{3}\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{11}{3}\\\frac{1}{3}\\0\\1\end{bmatrix}\right\}$

$\begin{bmatrix}1&2&1&3\\0&3&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow \begin{array}{l}x=-\frac{11}{3}\\y=\frac{1}{3}\end{array}$

(a)(iii)

$LUx=b\quad Ux=c\quad Lc=b$

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\-4\\2\end{bmatrix}\Rightarrow c=\begin{bmatrix}-2\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&2&1&3\\0&3&1&-1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\0\\0\end{bmatrix}=\begin{bmatrix}-2\\0\\0\end{bmatrix}\Rightarrow x_0=\begin{bmatrix}-2\\0\\0\\0\end{bmatrix}$$

GENERAL SOLUTION: $\begin{bmatrix}-2\\0\\0\\0\end{bmatrix}+t\begin{bmatrix}-\frac{1}{3}\\-\frac{1}{3}\\1\\0\end{bmatrix}+k\begin{bmatrix}-\frac{11}{3}\\\frac{1}{3}\\0\\1\end{bmatrix}$

Q5 (e)

THE PROJECTION MATRIX ONTO THE COLUMN SPACE OF A IS:

$$P = A\,(A^T A)^{-1} A^T$$

COMPUTE $A^T A$

$$A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$\det(A^T A) = 5 \neq 0 \Rightarrow A$ IS INVERTIBLE

$$(A^T A)^{-1} = \frac{1}{\det A} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ \frac{2}{5} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{p} = P\underline{e} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{4}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix}$$

DISTANCE FROM $\underline{e}$ IS $d = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5} - \frac{5}{5}\right)^2 + (0-0)^2} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}}$

Q5 (c)

n X n MATRIX IS DIAGONALIZABLE IFF IT HAS n DISTINCT EIGEN VECTORS.

$$A = \begin{vmatrix} a & b \\ 0 & c \end{vmatrix}$$

IF $a \neq c$ THEN EIGEN VECTORS ARE: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{b}{c-a} \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} \frac{b}{c-a} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{ab}{c-a} + \frac{b(c-a)}{c-a} \\ c \end{bmatrix} = c\begin{bmatrix} \frac{b}{c-a} \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \frac{b}{c-a} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & -\frac{b}{c-a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & \frac{bc}{c-a} \\ 0 & c \end{bmatrix}\begin{bmatrix} 1 & -\frac{b}{c-a} \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} a & \frac{-ab}{c-a} + \frac{bc}{c-a} \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

IF $a = c$ AND $b = 0$ THEN

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} aX \\ aY \end{bmatrix} = a\begin{bmatrix} X \\ Y \end{bmatrix}$$ HENCE $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ARE EIGEN-VECTORS.

$$\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

IF $a = c$ AND $b \neq 0$ THEN

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} aX + bY \\ aY \end{bmatrix}$$ NOTE $\begin{bmatrix} aX + bY \\ aY \end{bmatrix} = \lambda\begin{bmatrix} X \\ Y \end{bmatrix}$ IFF $Y = 0$

HENCE → ONLY ONE EIGEN VECTOR.

<u>ANSWER</u> A IS DIAGONALIZABLE WHEN $(a \neq c)$ OR $(a = c$ AND $b = 0)$

Q5 (d) (i)

$$A^TA = \begin{bmatrix} a & (1-a) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 1-a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2+(1-a)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

HENCE, EIGENVALUES ARE $a^2+(1-a)^2$ AND 0. (FOR ALL a)

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

SINGULAR VALUES $\sigma_1 = \sqrt{a^2+(1-a)^2}$

$\sigma_2 = \sigma_3 = 0$

$$\hat{q}_1 = \frac{A q_1^T}{\sqrt{a^2+(1-a)^2}} = \frac{\begin{bmatrix} a & 0 & 0 \\ 1-a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\sqrt{a^2+(1-a)^2}} = \begin{bmatrix} a \\ 1-a \\ 0 \end{bmatrix} \frac{1}{\sqrt{a^2+(1-a)^2}}$$

$$AA^T = \begin{bmatrix} a & 0 & 0 \\ 1-a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & 1-a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a^2 & (1-a)a & 0 \\ (1-a)a & (1-a)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EIGENVECTOR FOR $\lambda = \sqrt{a^2+(1-a)^2}$ IS $\begin{bmatrix} a \\ 1-a \\ 0 \end{bmatrix} \frac{1}{\sqrt{a^2+(1-a)^2}}$

IF $a \neq 0, a \neq 1$ THEN

ORTHOGONAL

EIGENVECTORS FOR $\lambda = 0$ ARE $\hat{q}_2 = \begin{bmatrix} 1 \\ -\frac{a}{1-a} \\ 0 \end{bmatrix} \frac{1}{\sqrt{1+\frac{a^2}{(1-a)^2}}}$ AND $\hat{q}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

IF $a=0$ THEN $\hat{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{q}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{q}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

IF $a=1$ THEN $\hat{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{q}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Q5 (d) (i) (CONTINUED)

SO WHEN $a \neq 1$

$$A = \begin{bmatrix} \frac{a}{\sqrt{a^2+(1-a)^2}} & \frac{1}{\sqrt{1+a^2/(1-a)^2}} & 0 \\ \frac{1-a}{\sqrt{a^2+(1-a)^2}} & -\frac{a}{1-a}\cdot\frac{1}{\sqrt{1+a^2/(1-a)^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{a^2+(1-a)^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{}_{Q}$$

IF $a = 1$ THEN

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) (ii)

IF $a = 1$ $\quad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$a \neq 1$ $\quad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{a^2+(1-a)^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{Q}^T$

Q6 (a)(i)

$H=T$ - TEST IS POSITIVE
$H=F$ - TEST IS NEGATIVE

$D=T$ - HAS DISEASE
$D=F$ - DOES NOT HAVE DISEASE

$P(H=F \mid D=F) = 0.92$
$P(H=T \mid D=T) = 0.95$
$P(D=T) = 0.01$

$$P(H=T) = P(H=T, D=F) + P(H=T, D=T) =$$
$$= P(H=T \mid D=F)\,P(D=F) + P(H=T \mid D=T)\,P(D=T) =$$
$$= (1-0.92)(1-0.01) + 0.95 \cdot 0.01 =$$
$$= 0.0792 + 0.0095 = 0.0887$$

(ii)

$$P(D=T \mid H=T) = \frac{P(D=T, H=T)}{P(H=T)} = \frac{P(H=T \mid D=T)\,P(D=T)}{P(H=T)}$$

$$= \frac{0.95 \cdot 0.01}{0.0887} \approx 0.108$$

Q6 (b)(i)

PROBABILITY OF k ERRORS IS

$$\binom{n}{k} p^k (1-p)^{n-k} = P(k \text{ ERRORS})$$

ERORS ARE UNDETECTED WHEN NUMBER OF ERRORS k IS EVEN AND >0.

SO FOR $n=3$, $p=0.2$ PROBABILITY

OF UNDETECTED ERROR IS

$$\binom{3}{2} p^2 (1-p)^{3-2} = \frac{3!}{2!1!} \cdot 0.032 = 0.096$$

IN GENERAL CASE

$$\sum_{k \text{ EVEN}, k>0} \binom{n}{k} p^k (1-p)^{n-k} = P(\text{UNDETECTED})$$

(ii) $P(\text{DETECTED}) = \sum_{k \text{ ODD}} \binom{n}{k} p^k (1-p)^{(n-k)}$

$$P(0 \text{ ERRORS}) + P(\text{DETECTED}) + P(\text{UNDETECTED}) = 1$$

FROM BINOMIAL THEOREM

$$(1-p-p)^n = \sum_k (-p)^k (1-p)^{n-k} \binom{n}{k} = \sum_{k \text{ EVEN}} p^k (1-p)^{n-k} - \sum_{k \text{ ODD}} p^k (1-p)^{n-k}$$

$$= P(0 \text{ ERROR}) + P(\text{UNDETECTED}) - P(\text{DETECTED})$$

$$P(\text{UNDETECTED}) = \frac{1+(1-2p)^n}{2} - (1-p)^n$$

Q6 (c) (i)

$X \sim \text{BINOMIAL}(n, p)$

$Y \sim \text{BINOMIAL}(m, p)$

$g_X(z) = (1 - p + pz)^n$ PGF OF X

$g_Y(z) = (1 - p + pz)^m$ PGF OF Y

$g_Z(z) = g_{X+Y}(z) = g_X(z) g_Y(z) = (1 - p + pz)^{n+m}$

THIS IS A PGF OF BINOMIAL $(n+m, p)$.

(ii)

$F_{Z|X}(z|x) = P(Z \leq z \mid X = x) = P(X + Y \leq z \mid X = x) =$

$= P(x + Y \leq z \mid X = x) \underset{①}{=} P(x + Y \leq z) = P(Y \leq z - x) = F_Y(z-x)$

① SINCE X AND Y ARE INDEPENDENT

$$\frac{\partial F_{Z|X}(z|x)}{\partial z} = f_{Z|X}(z|x)$$

$$\frac{\partial F_Y(z-x)}{\partial z} = f_Y(z-x)$$

HENCE $f_{Z|X}(z|x) = f_Y(z-x)$

Q6 (c)(iii)

$$f_{X|Z}(x|z) = \frac{f_{Z|X}(z|x)\, f_X(x)}{f_Z(z)} = \frac{f_Y(z-x)\, f_X(x)}{f_Z(z)} =$$

$$= \frac{1}{f_Z(z)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-z)^2}{2}} \propto e^{-\frac{x^2+x^2-2xz+z^2}{2}} \propto$$

$$\propto e^{\frac{-2\left(x-\frac{z}{2}\right)^2}{2}}$$

DISTRIBUTED AS $N\left(\mu = \frac{z}{2}, \sigma^2 = \frac{1}{2}\right)$.

(iv) $\perp$ $X \sim POIS(\lambda)$, $Y \sim POIS(\lambda)$

PGF $X \rightarrow g_X(z) = e^{\lambda(z-1)}$

PGF $Y \rightarrow g_Y(z) = e^{\lambda(z-1)}$

PGF $X+Y=Z \rightarrow g_Z(z) = g_X(z) g_Y(z) = e^{2\lambda(z-1)}$

HENCE $Z \sim POIS(2\lambda)$

$$E\left(\frac{1}{Z+1}\right) = \sum_{k=0}^{\infty} \frac{1}{k+1} e^{-2\lambda} \frac{(2\lambda)^k}{k!} = e^{-2\lambda} \sum_{k=0}^{\infty} \frac{(2\lambda)^k}{(k+1)!} = \frac{e^{-2\lambda}}{2\lambda} \sum_{k=0}^{\infty} \frac{(2\lambda)^{k+1}}{(k+1)!} \stackrel{1}{=}$$

$$= \frac{e^{-2\lambda}}{2\lambda}\left(e^{2\lambda} - 1\right) = \frac{1}{2\lambda}\left(1 - e^{-2\lambda}\right)$$

1 FROM TAYLOR SERIES OF e^{λ}.