Version EM2

EGT1
ENGINEERING TRIPOS PART IB

Friday 10 June 2022 14.00 to 16.10

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

*Answer not more than **two** questions from this section.*

1 Consider the functions $f(x, y) = e^{-(x^2+y^2)}$ and $g(x, y) = y/\sqrt{x^2 + y^2}$.

(a) Evaluate

$$\iint f(x, y)\, g(x, y)\, dx\, dy$$

over the region $\mathfrak{R}$ bounded by the curve $x^2 + y^2 = R^2$, the x-axis, and the straight line $x = y$. [10]

(b) Sketch the volume enclosed by the surface $z = f(x, y)$ and the two planes $z = 0$ and $z = e^{-2}$. [5]

(c) Calculate the volume of part (b). [10]

2 A two-dimensional incompressible fluid flow has velocity $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}$, with $u_x = Ax$, $u_y = -Ay$, and A a positive constant.

(a) Derive an equation for the field lines of **u** and sketch them. [5]

(b) Show that **u** is solenoidal and irrotational. [5]

(c) Evaluate the line integral

$$\int 2\mathbf{u} \cdot d\mathbf{l}$$

along the circular arc of unity radius, centred at the origin, between the points $(0, 1)$ and $(1, 0)$. [7]

(d) At time $t = 0$, a square region defined by the set of coordinates

$$A : (X_0, Y_0);\;\; B : (X_0 + L, Y_0);\;\; C : (X_0 + L, Y_0 - L);\;\; D : (X_0, Y_0 - L)$$

marks some of the fluid. Note that $L < Y_0$ and $L < X_0$. The vector **n** is normal to the sides of this region, therefore $\int_S \mathbf{u} \cdot \mathbf{n}\; dS$ is the flux of fluid crossing S, the perimeter of the region. As time evolves and the marked fluid moves with the flow, the marked region changes shape.

(i) Given that a fluid particle moving with the flow obeys $d\mathbf{X}/dt = \mathbf{u}$, where **X** is its position vector and t is time, show that the distance AB increases exponentially with t. Derive an expression for the distance BC with time. [4]

(ii) Considering your results from part (d)(i), or otherwise, describe how the area of the marked region changes with time. [4]

3 The scalar C represents the concentration of a contaminant diffusing in the x-direction. It obeys the partial differential equation (PDE)

$$\frac{\partial C}{\partial t} = \alpha \frac{\partial^2 C}{\partial x^2}$$

where α is a positive constant and t is time. The boundary conditions are $C(t,x) = 0$ at $x = \pm\infty$. At $t = 0$, the contaminant is concentrated at $x = 0$ so that $\int_{-\infty}^{\infty} C(0,x)dx = M$, with M a positive constant. It is also known that $\int_{-\infty}^{\infty} C(t,x)dx = M$ for all times t. We seek self-similar solutions for C for $t > 0$, so that $C(t,x) = g(t)f(\eta)$, where $x = \eta\sqrt{4\alpha t}$ and $g(t) = M/\sqrt{4\alpha t}$.

(a) Develop expressions for $\partial\eta/\partial t$ and $\partial\eta/\partial x$. [6]

(b) Show that the above PDE can be reduced to the ordinary differential equation

$$\frac{d^2 f}{d\eta^2} + 2\eta\frac{df}{d\eta} + 2f = 0$$

[12]

(c) Show that

$$C(t,x) = \frac{M}{\sqrt{4\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right)$$

is a solution to the above PDE. [7]

SECTION B

*Answer not more than **two** questions from this section*

4 (a)

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & k \\ k & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Solve $\mathbf{Ax} = \mathbf{b}$ for all values of k for which solutions exist. [6]

(b) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 3 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} c \\ d \\ e \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

(i) Find a QR decomposition of $\mathbf{A}$, making clear your intermediate calculation steps. [4]

(ii) Using the obtained QR decomposition, find the least squares solution for $\mathbf{Ax} = \mathbf{b}$. [3]

(c) Let $\mathbf{A} = \mathbf{bb}^\top + (a - b)\mathbf{I}$, where $\mathbf{b} = \begin{pmatrix} \sqrt{b} & \sqrt{b} & \cdots & \sqrt{b} \end{pmatrix}^\top$ is a $n \times 1$ column vector, with $b > 0$, and $\mathbf{I}$ is the $n \times n$ identity matrix.

(i) Find the eigenvalues, set of eigenvectors, and determinant of $\mathbf{A}$, for $n = 3$. [3]

(ii) Find the same properties of $\mathbf{A}$ as in part (c)(i), for the general case $n > 1$. [4]

(d) Consider a pair of similar matrices, $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ for some invertible matrix, $\mathbf{P}$.

(i) Show that $\mathbf{A}$ and $\mathbf{B}$ have the same determinant, eigenvalues and rank. [3]

(ii) Assume that a pair of matrices share all three properties mentioned in part (d)(i). Is it guaranteed that they are similar? [2]

5 (a) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 7 & 3 & 5 \\ -1 & 1 & 0 & -4 \end{pmatrix}$$

(i) Perform LU decomposition of the matrix $\mathbf{A}$, making clear your intermediate calculation steps. [4]

(ii) Identify the basis for each of the four fundamental subspaces of the matrix $\mathbf{A}$. [2]

(iii) Find the general solution of the equation $\mathbf{A}\mathbf{x} = \begin{pmatrix} -2 & -4 & 2 \end{pmatrix}^\top$. [2]

(b) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Find the projection matrix $\mathbf{P}$ corresponding to the projection onto the column space of $\mathbf{A}$, the projection $\mathbf{p}$ of $\mathbf{b}$, and its distance from $\mathbf{b}$. Make your intermediate calculation steps clear. [4]

(c) Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

Find all values of $a, b, c \in R$ for which matrix $\mathbf{A}$ is diagonalisable and provide corresponding diagonalisations. [6]

(d) Let

$$\mathbf{A} = \begin{pmatrix} a & 0 & 0 \\ 1-a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(i) Find a singular value decomposition of matrix $\mathbf{A}$, for all real values of a, making clear your intermediate calculation steps. [5]

(ii) Find a singular value decomposition of $\mathbf{A}^\top$. [2]

6 (a) A new test for a disease has been produced by a company. Trials revealed that if a test user has the disease, the test is positive with 95% probability. Similarly, if the user does not have the disease, the test is negative with 92% probability. Approximately 1% of the population is suspected to have this disease. Compute the probabilities below.

(i) The probability that a randomly chosen individual will test positive. [3]

(ii) The probability that a randomly chosen individual who tests positive will have the disease. [3]

(b) A sequence $x_1, x_2, \cdots, x_n$ of data bits ($x_i \in \{0, 1\}$), is transmitted over a noisy channel. The received sequence of bits is $Y_1, Y_2, \cdots, Y_n$ where $Y_i = x_i$, if there is no error in that bit and $Y_i = 1 - x_i$, otherwise. The error events are independent and any individual bit can be corrupted with the probability, p, where $p \in (0, 0.5)$.
To facilitate error detection, the nth bit is reserved for a parity check: $x_n = 0$, if $\sum_{i=1}^{n-1} x_i$ is even, otherwise $x_n = 1$. The existence of error in the transmission can be detected if Y_n has a different parity than $\sum_{i=1}^{n-1} Y_i$. If the parities are the same, either there has been no error in the transmission, or the number of corrupted bits is even and the existence of errors cannot be detected using this procedure.

(i) Write down the expression, as a sum, for the probability that the received message has errors which go undetected. Evaluate this expression for $n = 3$, $p = 0.2$. [4]

(ii) Give a simplified expression, not involving a sum of a large number of terms, for the same probability mentioned in Part (b)(i). [Hint: use the Binomial expansion

$$(a + b)^n = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$$

with $a = -p$ and $b = 1 - p$.] [3]

(c) Let X, Y be independent random variables. Also let $Z = X + Y$.

(i) Using probability generating functions, show that if $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$, then $Z \sim \text{Binomial}(n + m, p)$. [3]

(ii) Assume that X and Y are continuous and their probability density functions are $f_X(x)$ and $f_Y(y)$, respectively. By considering the cumulative density function of Z given X, or otherwise, show that the following holds: $f_{Z|X}(z|x) = f_Y(z - x)$. [3]

(iii) Derive the conditional probability density function of X, given that $Z = z$, when $X \sim \mathcal{N}(0, 1)$ and $Y \sim \mathcal{N}(0, 1)$. [3]

(iv) Compute the expectation of $1/(Z + 1)$, when $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\lambda)$. [3]

END OF PAPER