Q1

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x}$$: motion of taut string

(a) If $\phi = \phi_1$ and $\phi = \phi_2$ are both solutions of the PDE, the equation is linear if $\phi = A\phi_1 + B\phi_2$ is also a solution.

If ϕ_1 is a solution: $\frac{\partial^2 \phi_1}{\partial t^2} = c^2 \frac{\partial^2 \phi_1}{\partial x^2}$ (1)

" ϕ_2 " : $\frac{\partial^2 \phi_2}{\partial t^2} = c^2 \frac{\partial^2 \phi_2}{\partial x^2}$ (2)

(1) × A + (2) × B :
$$\frac{\partial^2}{\partial t^2}(A\phi_1 + B\phi_2) = A\frac{\partial^2 \phi_1}{\partial t^2} + B\frac{\partial^2 \phi_2}{\partial t^2}$$
$$\overset{(1),(2)}{=} c^2 A \frac{\partial^2 \phi}{\partial x^2} + c^2 B \frac{\partial^2 \phi_2}{\partial x^2}$$
$$= c^2 \frac{\partial^2 (A\phi_1 + B\phi_2)}{\partial x^2}$$ ∴ linear

(b) $\phi(x,t) = f(\alpha t + \beta x)$

Check solution by differentiation.

Assume $\xi = ct + \beta x$

$$\frac{\partial \phi}{\partial x} = \frac{df}{d\xi}\frac{\partial \xi}{\partial x} = \beta \frac{df}{d\xi}$$

$$\frac{\partial \phi}{\partial t} = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} = \alpha \frac{df}{d\xi}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \beta^2 \frac{d^2 f}{d\xi^2} = \beta^2 \frac{d^2 f}{d\xi^2}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial t^2} = \alpha^2 \frac{d^2 f}{d\xi^2} = \alpha^2 \frac{d^2 f}{d\xi^2}$$

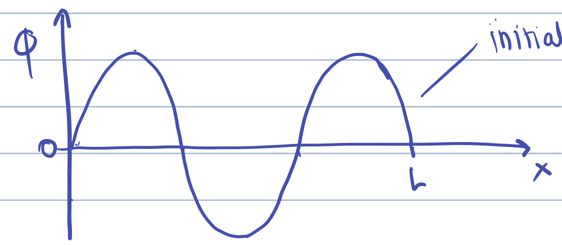
$$\therefore \frac{\partial^2 \phi}{\partial t^2} = \frac{\alpha^2}{\beta^2}\frac{\partial^2 \phi}{\partial x^2}$$

∴ it is a solution if $\frac{\alpha^2}{\beta^2} = c^2 \Rightarrow \boxed{\alpha = \pm c\beta}$

(c) $\phi(x,t) = f(\pm\beta ct + \beta x) \Rightarrow \phi = f(\beta ct + \beta x) + f(-\beta ct + \beta x)$ ← follows from linearity

$= \underbrace{f(\beta(ct+x))}_{\text{wave travelling in the -ve x direction}} + \underbrace{f(-\beta(ct-x))}_{\text{wave travelling in the +ve x direction}}$ → represents two waves travelling in opposite directions

(d)



Separation of variables: $\phi(x,t) = X(x)\,T(t)$

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} \Rightarrow X\frac{d^2T}{dt^2} = c^2\,T\frac{d^2X}{dx^2} \Rightarrow \frac{1}{Tc^2}\frac{d^2T}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \underset{-\omega^2}{\text{constant}}$$

$$\frac{1}{Tc^2}\frac{d^2T}{dt^2} = -\omega^2 \Rightarrow T(t) = A\cos(\omega ct) + B\sin(\omega ct)$$ A,B,C,D : constants

$$\frac{1}{X}\frac{d^2X}{dt^2} = -\omega^2 \Rightarrow X(x) = C\cos(\omega x) + D\sin(\omega x)$$

$$\phi(x,t) = X(x)\cdot T(t) \Rightarrow \phi(x,t) = (C\cos(\omega x) + D\sin(\omega x))(A\cos(\omega ct) + B\sin(\omega ct))$$

Using boundary conditions: @ $x=0$, $\phi=0\ \forall t \geq 0 \Rightarrow C=0$

@ $t=0$, string is released from rest

$\Rightarrow \frac{\partial\phi}{\partial t} = 0$ for all x

$$\Rightarrow \frac{\partial\phi}{\partial t} = D\sin(\omega x)(-A\omega c\sin(\omega ct) + B\omega c\cos(\omega ct))$$

$$= D\omega c\sin(\omega x)(-A\sin(\omega ct) + B\cos(\omega ct))$$

$\Rightarrow B = 0$

@ $x = L$, $\phi = 0 \ \forall t \geq 0$

$$\Rightarrow D \sin(\omega L) \, A \cos(\omega c t) = 0$$

$$\Rightarrow \omega L = n\pi \quad (\text{for } n = \text{integer } 1, 2, 3 \ldots)$$

$$\Rightarrow \omega = \frac{n\pi}{L}$$

@ $t = 0$, $\phi(x,0) = \sin \frac{3\pi x}{L}$

$\hookrightarrow$ we are looking at the 3^{rd} harmonic

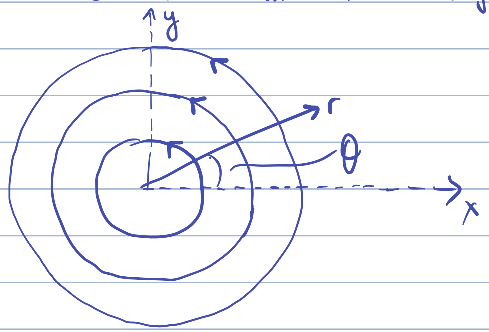
$\therefore$ the solution is : $\boxed{\phi(x,t) = \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{3\pi c}{L} t\right)}$

Q2

(a) $\underline{V} = \frac{A}{r}\underline{e_\theta}$ (A: constant)

(i) $\underline{V}$ independent of θ & constant for a given r

=> field lines : circles with centre the origin



(ii)

$$\nabla \times \underline{V} = \frac{1}{r}\begin{vmatrix} \underline{e_r} & r\underline{e_\theta} & \underline{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & r\frac{A}{r} & 0 \end{vmatrix} = 0\,\underline{e_r} + 0\,r\underline{e_\theta} + \frac{\partial}{\partial r}(A)\,\underline{e_z}$$

$$= \boxed{0}$$

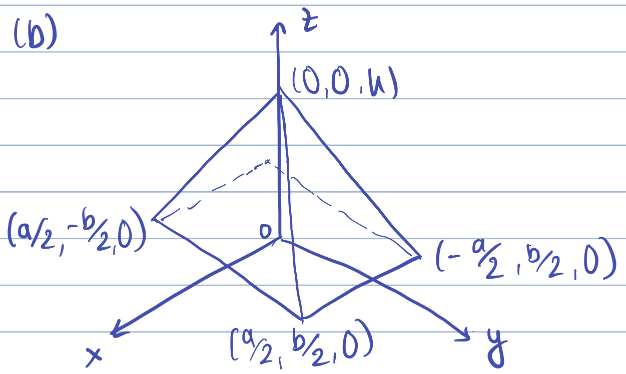
$\nabla \times \underline{V}$ is the vorticity:

The local magnitude of the curl of the velocity field is equal to twice the instantaneous mean angular velocity of a fluid particle at that point. The curl vector points in the direction of the axis of rotation of the fluid particle

(iii) The field is irrotational since $\nabla \times \underline{V} = 0$

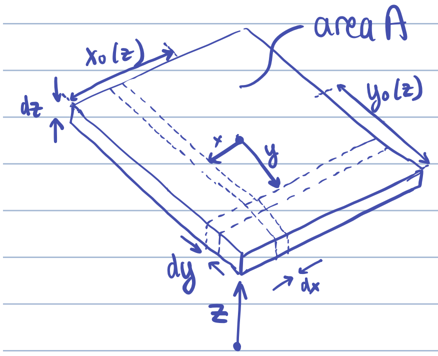
(iv) $\oint \underline{V}\cdot d\underline{r} = \iint \underbrace{\nabla \times \underline{V}}_{=0 \text{ from (ii)}} \cdot d\underline{A}$ Stokes's theorem

(b)



ii)

Infinitesimal volume element at point (x, y, z): $dV = dx\,dy\,dz$



$$V = \int_0^h \int_{-y_0(z)}^{+y_0(z)} \int_{-x_0(z)}^{+x_0(z)} dx\,dy\,dz$$

Finding expressions for $y_0(z)$ & $x_0(z)$:

$$\frac{y_0(z)}{b/2} = \frac{h-z}{h} \Rightarrow y_0(z) = \frac{b}{2h}(h-z)$$

$$\frac{x_0(z)}{a/2} = \frac{h-z}{h} \Rightarrow x_0(z) = \frac{a}{2h}(h-z)$$

$$\Rightarrow V = \int_0^h \int_{-\frac{b}{2h}(h-z)}^{+\frac{b}{2h}(h-z)} \int_{-\frac{a}{2h}(h-z)}^{+\frac{a}{2h}(h-z)} dx\,dy\,dz$$

$$= \int_0^h \left[\int_{-\frac{b}{2h}(h-z)}^{+\frac{b}{2h}(h-z)} dy \int_{-\frac{a}{2h}(h-z)}^{+\frac{a}{2h}(h-z)} dx \right] dz$$

$$= \int_0^h \left(\frac{b}{h}(h-z)\ \frac{a}{h}(h-z) \right) dz$$

$$= \int_0^h \frac{ab}{h^2}(h-z)^2\,dz$$

$$= -\frac{1}{3}\frac{ab}{h^2}(0-h^3)$$

$$= \boxed{\frac{abh}{3}}$$

(ii) Centre of mass (x_c, y_c, z_c) will lie on the z-axis due to symmetry

$\Rightarrow x_c = 0, \quad y_c = 0$

z_c should satisfy the following (assuming uniform density)

$$z_c V = \int_0^h z \cdot \underbrace{A}_{\text{area of pyramid slice at height } z} \, dz$$

from (i) $A = 2y_0(z)\, 2x_0(z) = \frac{b}{h}(h-z)\frac{a}{h}(h-z)$

$$\therefore z_c V = \int_0^h z\, ab \frac{(h-z)^2}{h^2} \, dz$$

$$\Rightarrow z_c V = \frac{ab}{h^2}\int_0^h (h^2 z - 2hz^2 + z^3)\, dz$$

$$\Rightarrow z_c V = \frac{ab}{h^2}\left[\frac{1}{2}h^2z^2 - \frac{2}{3}hz^3 + \frac{1}{4}z^4\right]_0^h$$

$$\Rightarrow z_c V = \frac{ab}{h^2}\left(\frac{1}{2}h^4 - \frac{2}{3}h^4 + \frac{1}{4}h^4\right)$$

$$\Rightarrow z_c V = \frac{abh^2}{12}$$

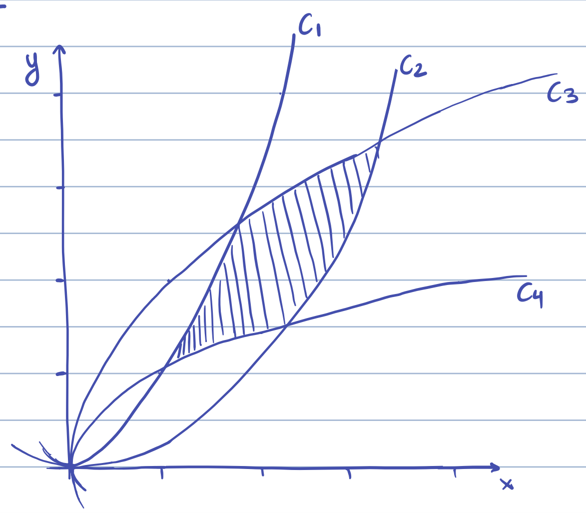
$$\overset{(i)}{\Rightarrow} z_c \frac{abh}{3} = \frac{abh^2}{12}$$

$$\Rightarrow \boxed{z_c = \frac{h}{4}}$$

So the centre of mass is at $(0, 0, h/4)$

Q3

(a)



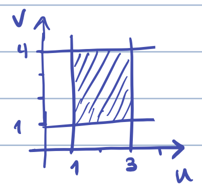
$x^2 y^{-1} = 1 \longrightarrow$ curve C_1

$x^2 y^{-1} = 3 \longrightarrow C_2$

$y^2 x^{-1} = 1 \longrightarrow C_3$

$y^2 x^{-1} = 4 \longrightarrow C_4$

(b) Required area : $\iint dx\,dy$



Use the transformation : $u = x^2 y^{-1}$

$v = y^2 x^{-1}$

$\Rightarrow$ Area becomes: $\displaystyle\int_1^4 \int_1^3 \frac{1}{J}\, du\, dv$ ← Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$= \frac{2\cancel{x}}{\cancel{y}} \frac{2\cancel{y}}{\cancel{x}} - \frac{\cancel{y^2}}{\cancel{x^2}} \frac{\cancel{x^2}}{\cancel{y^2}}$$

$$= 3$$

Therefore the area is $\displaystyle\int_1^4 \int_1^3 \frac{1}{3}\, du\, dv = \frac{1}{3} \left[u\right]_1^3 \left[v\right]_1^4$

$$= \frac{1}{3}(3-1)(4-1)$$

$$= \boxed{2}$$

(c)

$$\oint \underline{F} \cdot d\underline{l} = \iint_R \nabla \times \underline{F} \, dA \qquad \text{Stokes's theorem}$$

$$\underline{F} = xy^2 \, \underline{i} + 2x^2 y \, \underline{j}$$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2y & 0 \end{vmatrix} = \frac{\partial}{\partial x}\left(2x^2y\right) - \frac{\partial}{\partial y}\left(xy^2\right)$$

$$= 4xy - 2xy$$

$$= 2xy$$

Using the transformation of part (b) $\left[2xy = 2uv\right]$

$$\iint_R \nabla \times \underline{F} \, dA = \int_1^4 \int_1^3 \frac{1}{3} \, 2uv \, du \, dv$$

$$= \frac{2}{3} \left[\frac{1}{2} u^2\right]_1^3 \left[\frac{1}{2} v^2\right]_1^4$$

$$= \boxed{20}$$

2P7 - Draft exam 17/1/2025 - Crib. Shreyas Mandre

Section B

Q4. $\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & a \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 1 \\ 3 \\ b \end{bmatrix}$

(a) $\det(\underline{\underline{A}}) = 1\,(2a - (3)(5)) - 1\,((1a) - (3)(3)) + 1\,((1)(5) - (2)(3))$

$= 1\,(2a - 15) - 1\,(a - 9) + 1\,(5-6) = 2a - 15 - a + 9 - 1$

$= a - 7$

$\underline{\underline{A}}$ is not full rank $\Rightarrow \det(\underline{\underline{A}}) = 0 \Rightarrow a - 7 = 0 \Rightarrow \boxed{a = 7}$

(b) Use Gaussian elimination

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & a \end{bmatrix}$$

$R_2 - R_1$
$R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & a-3 \end{bmatrix}$$

$R_3 - 2R_2$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}_{\underline{\underline{L}}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & a-7 \end{bmatrix}}_{\underline{\underline{U}}}$$

Since $a=7$, the last row of $\underline{\underline{U}}$ is $\underline{0}$.

(c)

(i) $\det(\underline{\underline{A}}^4) = [\det(\underline{\underline{A}})]^4 = 0^4 = 0.$

(ii) $\text{Rank}(\underline{\underline{L}}) = 3$ and $\text{Rank}(\underline{\underline{U}}) = 2.$

(iii) For a solution to exist, $\underline{b}$ must be in the columnspace of $\underline{\underline{A}}$.

We know $\underline{\underline{A}}$ is singular, so column space is two-dimensional. Let us take the first & third columns as the basis.

$$\begin{bmatrix}1\\1\\3\end{bmatrix}\alpha + \begin{bmatrix}1\\3\\7\end{bmatrix}\beta = \begin{bmatrix}1\\3\\b\end{bmatrix} \Rightarrow \left.\begin{array}{l}\alpha+\beta=1\\ \alpha+3\beta=3\end{array}\right\} \underbrace{\alpha=0,\ \beta=1}$$

$$3\alpha+7\beta=b \qquad \Downarrow \qquad \boxed{b=7}$$

(iv) The particular solution is $\begin{bmatrix}0\\0\\1\end{bmatrix} \longleftarrow$ from part (e).

To find the general solution, we need the homogeneous sol^n.

$$\begin{bmatrix}1\\1\\3\end{bmatrix}c_1 + \begin{bmatrix}1\\2\\5\end{bmatrix}c_2 + \begin{bmatrix}1\\3\\7\end{bmatrix}c_3 = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{ll} c_1+c_2+c_3=0 & \ldots (E1)\\ c_1+2c_2+3c_3=0 & \ldots (E2)\\ 3c_1+5c_2+7c_3=0 & \ldots (E3)\end{array}$$

$\Rightarrow \quad c_2+2c_3=0 \quad \Rightarrow \quad c_2=-2c_3 \qquad (E2-E1)$

$\Rightarrow \quad c_1-c_3=0 \quad \Rightarrow \quad c_1=c_3 \qquad$ (Substitute in E1)

$\Rightarrow$ solution is $\quad c_1\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

General solution $\quad \underline{x} = \begin{bmatrix}0\\0\\1\end{bmatrix} + c_1\begin{bmatrix}1\\-2\\1\end{bmatrix}$ for any c_1.

Q5.

(a)

(i) a, b, c are determined using $P_s + P_c + P_r = 1$ so, for day $n+1$

If Day n is sunny: $\frac{1}{4} + \frac{1}{2} + a = 1 \Rightarrow a = \frac{1}{4}$

If Day n is cloudy: $\frac{1}{6} + \frac{2}{3} + b = 1 \Rightarrow b = \frac{1}{6}$.

If Day n is rainy: $\frac{1}{2} + \frac{1}{6} + c = 1 \Rightarrow c = \frac{1}{3}$.

(ii)
$$P_s(n+1) = P_{s|s} P_s(n) + P_{s|c} P_c(n) + P_{s|r} P_r(n)$$
$$= \frac{1}{4} x_1 + \frac{1}{6} x_2 + \frac{1}{2} x_3$$

(b)

(i) Since the probabilities in each row of $\underline{\underline{M}}$ must add up to 1.
$$\sum_{j=1}^{3} M_{ij} = 1.$$

To see that 1 is an eigenvalue, we use:

(i) Eigenvalues of $\underline{\underline{M}}$ and $\underline{\underline{M}}^T$ are identical.

(ii) $$\sum_{j=1}^{3} M_{ij} x_i = x_j \qquad \text{where } \underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

i.e. $\underline{x}\,\underline{\underline{M}} = 1\,\underline{x}$ so $\underline{x}$ is a left eigenvector with eigenvalue 1.

Therefore, 1 is an eigenvalue. To find right eigenvector:
$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow$$
Probabilities on day $n+1$ = Probabilities on day n.

$$\Rightarrow \left.\begin{aligned} \tfrac{1}{4}v_1 + \tfrac{1}{6}v_2 + \tfrac{1}{2}v_3 &= v_1 \\ \tfrac{1}{2}v_1 + \tfrac{2}{3}v_2 + \tfrac{1}{6}v_3 &= v_2 \\ \tfrac{1}{4}v_1 + \tfrac{1}{6}v_2 + \tfrac{1}{3}v_3 &= v_3 \end{aligned}\right\}$$

$-9v_1 + 2v_2 + 6v_3 = 0$

$3v_1 - 2v_2 + v_3 = 0 \Rightarrow v_3 = 2v_2 - 3v_1$

$3v_1 + 2v_2 - 8v_3 = 0$

$-9v_1 + 2v_2 + 6(2v_2 - 3v_1) = 0 \Rightarrow -27v_1 + 14v_2 = 0$

$3v_1 + 2v_2 - 8(2v_2 - 3v_1) = 0 \Rightarrow 27v_1 - 14v_2 = 0$

$\Rightarrow \boxed{v_1 = \frac{14 v_2}{27}}$, $v_3 = 2v_2 - 3 \cdot \frac{14}{27} v_2 = \left(2 - \frac{14}{9}\right) v_2$

$\Rightarrow \boxed{v_3 = \frac{4}{9} v_2,}\ \boxed{= \frac{12}{53}}$

Since $v_1 + v_2 + v_3 = 1$

$v_2 + \frac{14}{27} v_2 + \frac{12}{27} v_2 = 0$

$\Rightarrow \frac{53}{27} v_2 = 0 \Rightarrow \boxed{v_2 = \frac{27}{53}}$

$\boxed{v_1 = \frac{14}{53}}$

(ii) $\lambda_{2,3}$ satisfy $\quad 1+17\lambda-90\lambda^2+72\lambda^3=0$, $\lambda=1$ is a root

$$1-\lambda+18\lambda-18\lambda^2-72\lambda^2+72\lambda^3=0$$

$$(1-\lambda)\left[1+18\lambda-72\lambda^2\right]=0$$

$$(1-\lambda)\left[1+18\lambda+81\lambda^2-153\lambda^2\right]=0$$

Either $\lambda=1$ or $(1-9\lambda)^2=153\lambda^2 \Rightarrow 1-9\lambda=\pm\lambda\sqrt{153}$

$\Rightarrow 1=(9\pm\sqrt{153})\lambda$

$\Rightarrow \lambda_{2,3}=\dfrac{1}{9\pm\sqrt{153}}$.

(c)

$$\underline{x}_n = c_1\lambda_1^n \underline{v}_1 + c_2\lambda_2^n \underline{v}_2 + c_3\lambda_3^n \underline{v}_3$$

Because $|\lambda_2|$ and $|\lambda_3|<1$, so as $n\to\infty$, $\underline{x}_n \to c_1\lambda_1^n v_1$

$= v_1$.

Since there are 365 day in a year, and the fraction of sunny, cloudy & rainy are given by $\frac{14}{53}$, $\frac{27}{53}$ & $\frac{12}{53}$, respectively (components of v_1), we have.

sunny days expected $= \frac{14}{53}\times 365 = 96$ days

cloudy days expected $= \frac{27}{53}\times 365 = 186$ days

rainy days expected $= \frac{12}{53}\times 365 = 83$ days

Q6.

(a) The probability distribution $P(t) = 1 - e^{-rt}$.

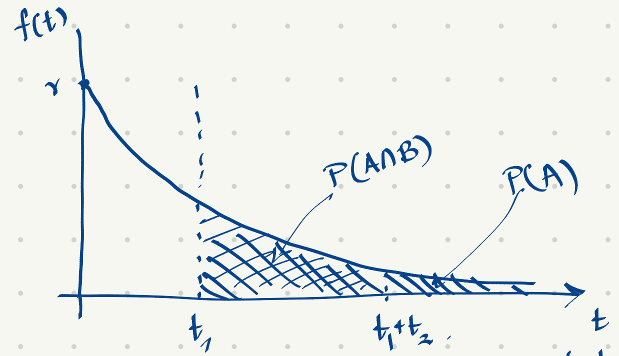
The density is $\frac{dP}{dt} = r e^{-rt} = f(t)$

(b) $E[t] = \int_0^\infty t f(t)\, dt = \frac{1}{r}\int_0^\infty t r e^{-rt} (r\,dt) = \frac{1}{r}\int_0^\infty x e^{-x} dx = \frac{1}{r}$.

(c) $MGF = \int_0^\infty f(t) e^{st}\, dt = \int_0^\infty r e^{-rt} e^{st}\, dt = \frac{r}{s-r} e^{(s-r)t}\Big|_0^\infty = \frac{r}{r-s}$.

$(s < r)$.

(d)



Let A = event bus does not arrive in time t_1

B = event bus arrives before time $t_1 + t_2$.

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\int_{t_1}^{t_1+t_2} f(t)\,dt}{\int_{t_1}^{\infty} f(t)\,dt} = \frac{\int_{t_1}^{t_1+t_2} r e^{-rt}\,dt}{\int_{t_1}^{\infty} r e^{-rt}\,dt}$$

$$= \frac{-e^{-rt}\Big|_{t_1}^{t_1+t_2}}{-e^{-rt}\Big|_{t_1}^{\infty}} = \frac{e^{-rt_1} - e^{-r(t_1+t_2)}}{e^{-rt_1} - 0} = 1 - e^{-rt_2}.$$

Therefore the density is $\frac{d}{dt}\left(1 - e^{-rt_2}\right) = r e^{-rt_2}$.

(e) $E(t_2) = \int_0^\infty t_2 r e^{-rt_2}\, dt_2 = \frac{1}{r}$ as before.

(f) Given n i.i.d. random variables $\underline{T} = \{T_1, T_2, \dots T_n\}$ the likelihood of a sample $\underline{t} = \{t_1, t_2, \dots t_n\}$ is

$$f_{\underline{T}|r}(\underline{t}|r) = f_{T_1|r}(t_1|r) \times f_{T_2|r}(t_2|r) \times \dots f_{T_n|r}(t_n|r)$$

$$= r^n \exp\{-r\sum_{i=1}^n t_i\} = r^n e^{-nr\bar{t}} \quad \text{where } \bar{t} = \frac{1}{n}\sum_{i=1}^n t_i$$

maximum when $n r^{n-1} e^{-nr\bar{t}} - n r^n \bar{t} e^{-nr\bar{t}} = 0 \Rightarrow \bar{t} r = 1$

We equate $t_1 + t_2 + t_3 \dots + t_{100} = 600$ min to get

$\boxed{r = \frac{1}{6\,\text{mins}}}$ as the maximum likelihood estimate for r.