Version SM/1

EGT1
ENGINEERING TRIPOS PART IB

Tuesday 17 June 2025 2 to 4.10 pm

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

SECTION A

*Answer not more than **two** questions from this section.*

1 Consider the vibrating motion of a taut string described by the following partial differential equation:

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

where $\phi(x, t)$ is the transverse displacement of the string, x is the spatial dimension along the string, t is time, and c is a constant.

(a) Show whether this partial differential equation is linear or non-linear. [6]

(b) Assuming that f is a function that can be differentiated twice, show that $\phi(x, t) = f(\alpha t + \beta x)$ is a solution to the partial differential equation and determine all possible relationships between α and β which satisfy the equation. [6]

(c) Taking into account your answer in part (a), explain how the solution in part (b) can be interpreted physically. [3]

(d) Assume that the string is secured at $x = 0$ and at $x = L$, and, given an initial displacement of shape $\sin\left(\frac{3\pi x}{L}\right)$, it is released from rest at $t = 0$. Using separation of variables, derive the solution for $t \geq 0$. [10]

2 (a) The velocity field of a 2-D flow in cylindrical polar coordinates is $\mathbf{V} = \frac{A}{r}\mathbf{e}_\theta$ where A is a constant.

(i) Sketch the field lines of this flow. [2]

(ii) Excluding the origin of the coordinate system, find $\nabla \times \mathbf{V}$ and explain its physical meaning. [5]

(iii) Is this flow field rotational or irrotational? [2]

(iv) Find $\oint \mathbf{V} \cdot d\mathbf{r}$ around the closed loop defined by the lines $x = 1$, $x = 2$, $y = 3$, and $y = 4$ in Cartesian coordinates. [2]

(b) A rectangular pyramid of uniform density has base lengths a and b, and apex h, as sketched in Figure 1.

(i) Evaluate the volume of the pyramid. [6]

(ii) Find the coordinates (x_c, y_c, z_c) of its centre of mass. [8]

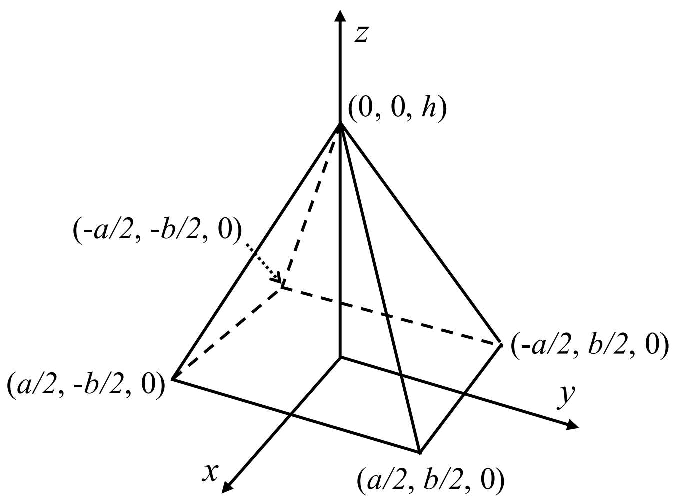

Figure 1

3 Consider the region $\mathfrak{R}$ bounded by the curves $x^2y^{-1} = 1$, $x^2y^{-1} = 3$, $y^2x^{-1} = 1$, and $y^2x^{-1} = 4$.

(a) Sketch $\mathfrak{R}$ for $x \geq 0$ and $y \geq 0$. [6]

(b) Evaluate the area of $\mathfrak{R}$. [9]

(c) If $d\mathbf{l}$ is the infinitesimal line element along the boundary of $\mathfrak{R}$, find $\oint \mathbf{F} \cdot d\mathbf{l}$ when $\mathbf{F} = xy^2\,\mathbf{i} + 2x^2y\,\mathbf{j}$. [10]

SECTION B

*Answer not more than **two** questions from this section.*

4 Consider the system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x}$ is a 3×1 matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & a \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ c \end{bmatrix}.$$

Here a and c are parameters.

(a) It is given that the rank of $\mathbf{A}$ is 2. Determine a and explain your reasoning. [3]

(b) Decompose $\mathbf{A} = \mathbf{LU}$ for a general value of a, where $\mathbf{U}$ is a 3×3 upper triangular matrix and $\mathbf{L}$ is a 3×3 lower-triangular one. [6]

(c) From now on, assume $a = 7$.

(i) Determine $\det(\mathbf{A}^4)$. [2]

(ii) State the ranks of $\mathbf{U}$ and $\mathbf{L}$. [2]

(iii) Determine the condition on c that a solution $\mathbf{x}$ exists. Explain your answer. [5]

(iv) Determine the general solution for the vector $\mathbf{x}$. [7]

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5 The weather in a town varies between being sunny, cloudy, and rainy.

(a) The probabilities p_s, p_c and p_r that on a given day the weather is sunny, cloudy or rainy, respectively, are conditional upon the weather on the previous day. If on day number n, the state of the weather is sunny, on day $n + 1$ the probabilities for a sunny, cloudy or rainy weather, respectively, are $p_{s|s} = 1/4$, $p_{c|s} = 1/2$ and $p_{r|s} = a$. Similarly, if it is cloudy on day n, $p_{s|c} = 1/6$, $p_{c|c} = 2/3$ and $p_{r|c} = b$, respectively, the next day. Finally, if it is rainy on day n, the probabilties for day $n + 1$ are $p_{s|r} = 1/2$, $p_{c|r} = 1/6$, and $p_{r|r} = c$.

(i) Determine a, b and c? State the principle you used to determine them. [2]

(ii) Suppose that on day n, $p_s = x_1$, $p_c = x_2$ and $p_r = x_3$, determine the probability that it will be sunny on day $n + 1$. State the principle you used to determine it. [3]

(b) Let

$$\mathbf{M} = \begin{bmatrix} 1/4 & 1/6 & 1/2 \\ 1/2 & 2/3 & 1/6 \\ 1/4 & 1/6 & 1/3 \end{bmatrix}.$$

(i) Noting that

$$\sum_{i=1}^{3} M_{ij} = 1, \qquad j = 1, 2, 3,$$

show that one of the eigenvalues of $\mathbf{M}$ is $\lambda_1 = 1$. Determine the corresponding eigenvector $\boldsymbol{v}_1$. [6]

(ii) It is given that the characteristic polynomial of $\mathbf{M}$ is

$$\mathbf{I} + 17\mathbf{M} - 90\mathbf{M}^2 + 72\mathbf{M}^3 = 0.$$

Determine the other two eigenvalues, λ_2 and λ_3, of $\mathbf{M}$. [7]

(c) Suppose that $\boldsymbol{x}(n) = [x_1(n), x_2(n), x_3(n)]^T$ denote the vector of probabilities of the three states (x_1=sunny, x_2=cloudy and x_3=rainy, respectively) on day n. Then the corresponding vector on day $n+1$ is given by $\boldsymbol{x}(n+1) = \mathbf{M}\boldsymbol{x}(n)$. Noting that both λ_2 and λ_3 are smaller in magnitude than unity, determine the limit of $\boldsymbol{x}(n)$ for $n \to \infty$. Based on this, determine the average number of days in a year that are sunny, cloudy and rainy, respectively. [7]

6 At a bus stop, the arrival time T between successive buses may be treated as a continuous random variable. The probability that the next bus *does not* arrive within time t is e^{-rt}, where r is a constant positive parameter.

(a) Determine the cumulative distribution function $F_T(t)$ and the probability density function $f_T(t)$ for T. [3]

(b) Determine the expected time between successive arrivals of buses. [3]

(c) Determine the moment generating function for the random variable T. [4]

(d) Alice narrowly misses a bus as she arrives at the stop. No bus arrives in time t as she waits. Consider the additional waiting time T' until a bus arrives as another continuous random variable. Find the probability density function for T'. [6]

(e) Determine the expected value of T'. [2]

(f) Bob patiently waits at the bus stop recording 100 inter-arrival times. The 101^{st} bus arrives 600 mins after the first. Using this information, estimate r using maximum likelihood. [7]

END OF PAPER

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