Version MPJ/3

EGT2
ENGINEERING TRIPOS PART IIA

Monday 22 April 2024 9.30 to 12.40

Module 3A1

FLUID MECHANICS I

*Answer not more than **five** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number __**not**__ your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachments:

- Incompressible Flow Data Card (2 pages);
- Boundary Layer Theory Data Card (1 page);
- 3A1 Data Sheet for Applications to External Flows (2 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The velocity potential for 2D small amplitude sloshing motion of an inviscid fluid with equilibrium depth h in a container of width πa, as shown in Fig. 1, is given by:

$$\phi(x, z, t) = \cosh\left(\frac{z+h}{a}\right) \sin\left(\frac{x}{a}\right) f(t)$$

where the function $f(t)$ will be derived later. The free surface is located at $z = \zeta(x, t)$, where

$$\zeta = A \sin\left(\frac{x}{a}\right) \sin(\omega t)$$

(a) Show that this velocity potential satisfies continuity, as well as the kinematic boundary conditions on the sides of the container. [20%]

(b) Sketch contours of the velocity potential $\phi(x, z)$ for $f(t) = 1$. Also sketch the corresponding streamlines. [30%]

(c) For small perturbations, the kinematic boundary condition at the surface is that the vertical velocity, w, satisfies $w = \partial\zeta/\partial t$, evaluated at $z = 0$. Use this boundary condition to derive an expression for $f(t)$. [10%]

(d) The dynamic boundary condition at the free surface is $\partial\phi/\partial t + g\zeta = 0$, where g is the acceleration due to gravity. This is derived from the unsteady Bernoulli equation. What does this boundary condition represent physically? By applying this boundary condition (evaluated at $z = 0$) to your previous answers, show that $\omega^2 = (g/a)\tanh(h/a)$. What does ω represent? What can you infer about how to transport a glass of water? [40%]

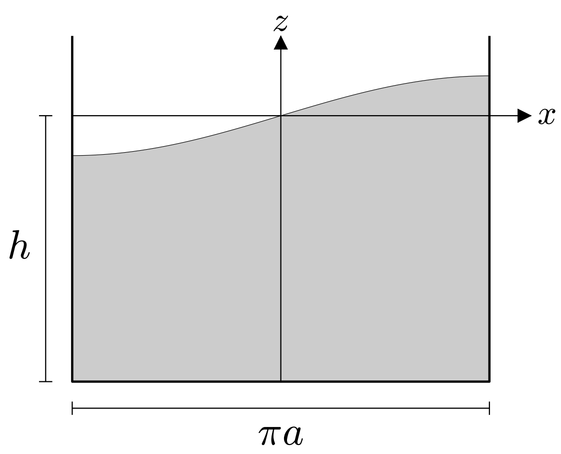

Fig. 1

2 (a) The streamfunction of an inviscid flow with free stream velocity U around a cylinder with radius a is:

$$\psi = U \sin\theta \left(r - \frac{a^2}{r} \right)$$

where (r, θ) are cylindrical polar coordinates. Show that this satisfies all the required boundary conditions. [20%]

(b) The free stream fluid is hot. In order to thermally protect the cylinder, cool fluid with the same density is forced radially through the surface of the cylinder. The bleed velocity, m, is uniform along the surface of the cylinder. Show that this can be modelled by a source at the origin and calculate the source strength. Show that the width of the cool fluid stream far downstream is $2\pi am/U$. [20%]

(c) Write down the streamfunction of this flow and calculate its value on the stagnation streamline far upstream. Sketch the streamlines of this flow when $\pi m/U < 1$. [20%]

(d) Define $\alpha = \pi - \theta$, such that α is the angle from the front of the cylinder. Derive an expression for the radial position of the stagnation streamline as a function of α. You may find it helpful to define $\lambda/a = (m/2U)(\alpha/\sin\alpha)$. [20%]

(e) What is the closest distance between the hot fluid and the cylinder when $m \ll 2U$? [20%]

3 For this question, you will need to use the *error function*, which is shown in Fig. 2. This is is proportional to the integral of a Gaussian:

$$\text{erf}\left(\frac{x}{2}\right) = \frac{1}{\sqrt{\pi}} \int_0^x \exp\left(-\frac{\xi^2}{4}\right) \, \mathrm{d}\xi$$

A semi-infinite stationary fluid is bounded by an infinitely long flat plate at $y = 0$. At $t = 0$, the plate starts to move in the x-direction with constant speed U.

(a) Starting from the Navier–Stokes equation,

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

show, with reasoning, that the $x-$velocity, u, satisfies:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$ [20%]

(b) By expressing $u(x, y)$ as a function of a similarity variable $\xi = y(\nu t)^{-1/2}$, derive the following ordinary differential equation for $u(\xi)$:

$$2u'' + \xi u' = 0$$ [40%]

(c) By applying suitable boundary conditions, derive a general expression for $u(y, t)$. [20%]

(d) At $t = T$, the plate stops moving. Explaining your reasoning, write down an expression for $u(y, t)$ for $t \geq T$. [20%]

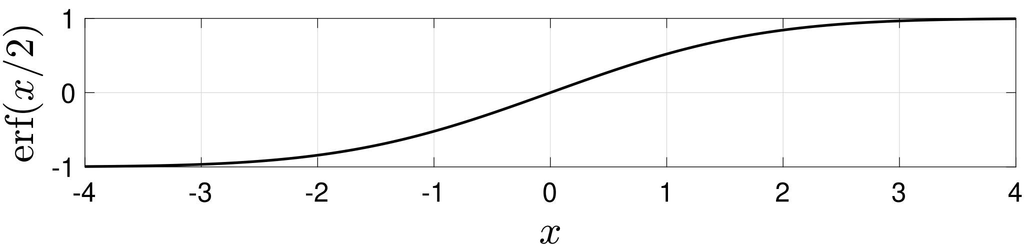

Fig. 2

4 Consider a steady incompressible flow in a rotational frame with a uniform angular velocity $\mathbf{\Omega} = (0,\ 0,\ \Omega)$ in the Cartesian coordinates $(x,\ y,\ z)$. The momentum equation relative to this frame is

$$\mathbf{u} \cdot \nabla \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

where $\mathbf{u}$ denotes the fluid velocity relative to the rotating frame and p the reduced pressure, which contains the centrifugal term. Suppose that the flow is composed of an inviscid free stream rotating about the z-axis, and a boundary layer near $z = 0$. Consider the case where the inertia term, $\mathbf{u} \cdot \nabla \mathbf{u}$, is negligible compared with the Coriolis term, $2\mathbf{\Omega} \times \mathbf{u}$.

(a) Let U and L denote typical scales of velocity and length of the flow. Under which condition is the inertia term negligible? [10%]

(b) The inviscid flow in the free stream is a uniform flow with velocity u_0 in the x-direction. Associated with u_0 is a pressure field p_0 with uniform pressure gradient ∇p_0. Calculate the components of this pressure gradient. [10%]

(c) The velocity components in the boundary layer are denoted (u, v, w). Use continuity to deduce that $w = 0$ for the special solution in which the velocity field depends only on z, *i.e.* $u = u(z)$ and $v = v(z)$. [10%]

(d) In the boundary layer, $\partial^2 \mathbf{u}/\partial x^2$ and $\partial^2 \mathbf{u}/\partial y^2$ are negligible compared with $\partial^2 \mathbf{u}/\partial z^2$. Write down the components of the momentum equation for u and v. [10%]

(e) In the boundary layer, the pressure is independent of z, hence $\partial p/\partial x = \partial p_0/\partial x$ and $\partial p/\partial y = \partial p_0/\partial y$. Deduce two ordinary differential equations for u and v, in which the pressure terms have been eliminated. [20%]

(f) Define a complex variable $f = u - u_0 + iv$. Deduce a second order equation for f and find the general solution for f. [30%]

(g) What are the boundary conditions for f at $z = 0$ and $z = +\infty$? Use these conditions to determine u and v. [10%]

5 Consider a fully developed turbulent flow in a circular pipe. Let ρ be the density of the fluid, ν be the kinematic viscosity and V be the average velocity in the pipe.

(a) The Darcy friction factor f is defined as

$$f = \frac{\mathrm{d}p/\mathrm{d}x}{\frac{1}{2}\rho V^2/D}$$

where $\mathrm{d}p/\mathrm{d}x$ is the pressure gradient and D is the diameter of the pipe. Express f in terms of the wall shear stress τ_w. [10%]

(b) Assume that the logarithmic-law of boundary layer theory can model the local mean velocity $u(r)$ all the way across a circular pipe of radius R with centreline at $r = 0$:

$$\frac{u(r)}{u^*} = \frac{1}{\kappa} \ln\left(\frac{u^*(R-r)}{\nu}\right) + B$$

where $u^* = \sqrt{\tau_w/\rho}$ is the friction velocity, and κ and B are two dimensionless constants. Deduce an equation between the mean velocity V and the friction velocity u^*. You may use

$$\int y \ln(y)\, \mathrm{d}y = \frac{1}{2}y^2 \ln(y) - \frac{1}{4}y^2$$ [40%]

(c) Establish a relation between the Darcy friction factor f and the Reynolds number $\mathrm{Re} = VD/\nu$. [20%]

(d) Calculate the ratio of the maximum velocity $u_{\max}$ to the mean velocity V in terms of f and κ. [20%]

(e) Describe the main differences between laminar and turbulent velocity profiles of flow in a pipe. [10%]

6 An aerofoil has the following camberline:

$$\frac{y}{c} = \frac{1}{5}\left[\frac{x}{c} - \left(\frac{x}{c}\right)^2\right] \qquad 0 \le \frac{x}{c} < 0.5$$

$$\frac{y}{c} = \frac{1}{10}\left[1 - \frac{x}{c}\right] \qquad 0.5 \le \frac{x}{c} < 1.0$$

(a) Use thin aerofoil theory to determine the zero-lift angle of attack (in degrees). [80%]

(b) At what angle of attack (in degrees) would you expect the maximum adverse pressure gradient along the aerofoil surface to be a minimum? Explain your answer with sketches. [20%]

7 An aircraft with weight W is in steady flight at speed V through air with density ρ. The aircraft has a wing with semi-span s and aspect ratio A_R.

(a) Determine the induced drag coefficient from lifting line theory as a function of W, V, ρ, s and A_R. You may assume that the wing loading is elliptical. [10%]

(b) Using the simple horseshoe vortex model, estimate the induced drag coefficient. Explain why this result differs from that of part (a). [30%]

(c) Show that a modified horseshoe vortex model that uses the effective span s' and an effective downwash angle w' improves the estimate of the induced drag coefficient. Express s' and w' as functions of W, V, ρ, and A_R. [30%]

(d) Compare the relationships between the induced drag and lift coefficients for (i) the lifting line model and (ii) the modified horseshoe vortex model used in (c). [20%]

(e) Which aspects of the wing flow are better described by the modified horseshoe vortex model, and which weaknesses remain? [10%]

8 (a) A passenger vehicle with a frontal area of 2.5 m^2 has a radiator with an effective open area of 0.1 m^2. While on a stationary test rig at a typical power output, the engine requires an average coolant flow rate of 0.05 $m^3 s^{-1}$ across the radiator.

(i) Using the momentum equation applied to a suitable control volume, estimate the percentage drag contribution due to this radiator while driving with the same power output. The effective velocity of the coolant air flow is assumed to be 20% of the vehicle speed. [25%]

(ii) How can the drag due to the radiator be reduced? How much of the drag calculated in (i) would you expect to remove at motorway speeds? [25%]

(b) Explain briefly, with the aid of sketches, how trailer side-skirts affect the drag of articulated heavy goods vehicles. [25%]

(c) Early road speed record vehicles often featured rear vertical fins. Explain, with the aid of sketches, why this was found to be necessary. [25%]

END OF PAPER

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1. (b) sketch cosh, then sin, then combine
1. (c) $f(t) = \omega A \cos(\omega t) / \left(\frac{1}{a} \sinh \frac{h}{a} \right)$
2. (a) $r \to \infty$ and $r \to a$
2. (b) $2\pi am$
2. (c) πam
2. (e) $m/(2U)$
3. (c) $U \left(1 - \text{erf} \left(\frac{y}{2\sqrt{\nu t}} \right) \right)$
4. (f) $2\Omega f + \mathrm{i}\nu f'' = 0$
5. (b) $V = u^* \left(B + \frac{1}{\kappa} \left(\ln \left(\frac{u^* R}{\nu} \right) - \frac{3}{2} \right) \right)$
5. (d) $\frac{u_{max}}{V} = 1 + \frac{3}{4\kappa} \left(\frac{f}{2} \right)^{1/2}$
6. (a) 5.47°
6. (b) 0.782°
8. (a) $\approx 5\%$