1. $\phi(x,z,t) = \cosh\left(\frac{z+h}{a}\right)\sin\left(\frac{x}{a}\right)f(t)$; $\zeta(x,t) = A\sin\left(\frac{x}{a}\right)\sin(\omega t)$

(a) $\underline{u} = \nabla\phi$ so $\nabla\cdot\underline{u} = \nabla^2\phi = 0$ by continuity. So check that $\nabla^2\phi = 0$ [1]

$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{1}{a^2}\phi + \frac{1}{a^2}\phi = 0$ for all values of x, z, and $f(t)$ [1]

Kinematic boundary condition on left and right boundary is $u = 0$ at $x = \pm\frac{\pi a}{2}$ $\forall z$

$u = \frac{\partial\phi}{\partial x} = \frac{1}{a}\cosh\left(\frac{z+h}{a}\right)\cos\left(\frac{x}{a}\right)f(t)$. When $x = \pm\frac{\pi a}{2}$, $\cos\left(\frac{x}{a}\right) = \cos\left(\pm\frac{\pi}{2}\right) = 0$ [1]

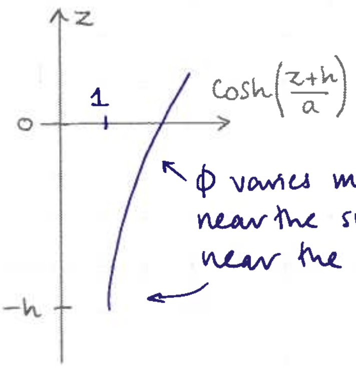
QED

Kinematic boundary condition on bottom boundary is $w = 0$ at $z = -h$ $\forall x$:

$w = \frac{\partial\phi}{\partial z} = \frac{1}{a}\sinh\left(\frac{z+h}{a}\right)\sin\left(\frac{x}{a}\right)f(t)$. When $z = -h$, $\sinh\left(\frac{z+h}{a}\right) = \sinh(0) = 0$ QED. [1]

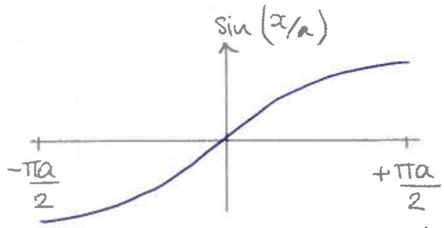
Most candidates completed this successfully, although several forgot the boundary condition at $z = -h$.

(b) First, sketch $\cosh\left(\frac{z+h}{a}\right) \rightarrow$



ϕ varies more rapidly near the surface than near the bottom

Then sketch $\sin\left(\frac{x}{a}\right) \rightarrow$



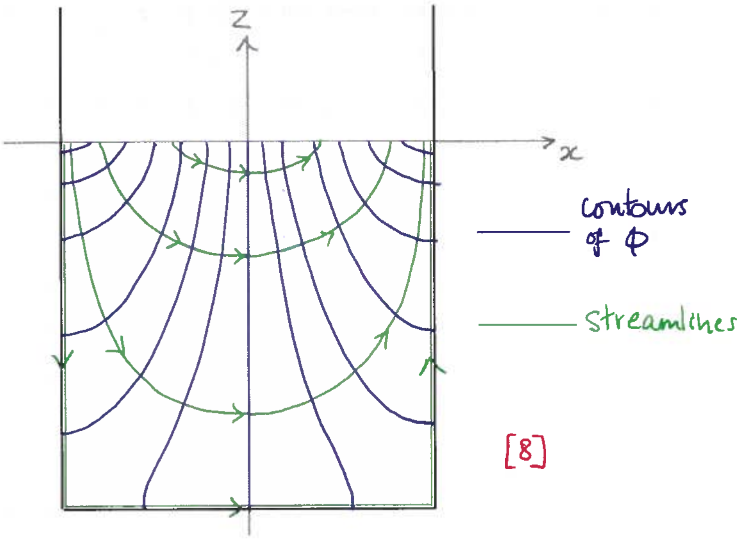
ϕ has the same x-dependence throughout the fluid

$u = \frac{\partial\phi}{\partial x} = 0$ on left and right walls
$\Rightarrow$ contours are perpendicular to left and right walls

$w = \frac{\partial\phi}{\partial z} = 0$ on bottom wall $\Rightarrow$ contours are perpendicular to bottom wall

$\phi = 0$ at $x = 0$ $\Rightarrow$ contour on z-axis

The streamlines are perpendicular to the velocity potential everywhere.



[8]

Only a few students completed this successfully. They did so by following the method shown above.

(c) $w = \frac{\partial\phi}{\partial z} = \frac{1}{a}\sinh\left(\frac{\cancel{z}^{0}+h}{a}\right)\sin\left(\frac{x}{a}\right)f(t) = \omega A\sin\left(\frac{x}{a}\right)\cos(\omega t)$

(at $z=0$)

$$\Rightarrow f(t) = \frac{\omega A\cos(\omega t)}{\frac{1}{a}\sinh\left(\frac{h}{a}\right)}$$ [2]

This was answered correctly by most candidates

(d) This boundary condition expresses the conservation of mechanical energy. The unsteady Bernoulli equation gives p as a function of ϕ. This is being evaluated at the free surface, which has height ζ. The dynamic boundary condition is set by imposing that $p = p_{atm}$ at the free surface $z=\zeta$. This leads to a balance between the force due to gravity (per unit mass) and the fluid acceleration.

$$\phi(x,z,t) = \cosh\left(\frac{z+h}{a}\right)\sin\left(\frac{x}{a}\right) f(t) \quad \text{where} \quad f(t) = \frac{\omega A \cos(\omega t)}{\frac{1}{a}\sinh\left(\frac{h}{a}\right)}$$

$$\Rightarrow \left.\frac{\partial \phi}{\partial t}\right|_{z=0} = -a\coth\left(\frac{h}{a}\right)\sin\left(\frac{x}{a}\right)\omega^2 A\sin(\omega t) = -g\zeta = -gA\sin\left(\frac{x}{a}\right)\sin(\omega t)$$

$$\Rightarrow \omega^2 = \frac{g}{a\coth\left(\frac{h}{a}\right)} = \frac{g}{a}\tanh\left(\frac{h}{a}\right)$$

The dynamic boundary condition is satisfied only when ω^2 has the form above. ω is the natural angular frequency of sloshing motion of a liquid in a container of depth h and width πa. You should avoid shaking the container at this frequency. [8]

For interest a 10 cm wide, 15 cm deep container has:

$$\omega^2 = \frac{10}{\frac{0.1}{\pi}\coth\left(\frac{0.15}{0.10}\pi\right)} = 314. \quad \Rightarrow \text{natural freq} = \frac{\omega}{2\pi} \simeq 2.8\text{ Hz}$$

A human's natural running frequency is around 3 Hz, so avoid running with a drink.

Around one third of the candidates answered this part reasonably well. Most had a good physical understanding of the motion, the meaning of ω, and the consequences for drink-carrying.

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2.

(a) $\psi = U \sin\theta \left(r - \frac{a^2}{r} \right)$ As $r \to \infty$, $\psi \to U r \sin\theta = Uy$, which is the streamfunction for uniform flow in the x-direction, as required. [2]

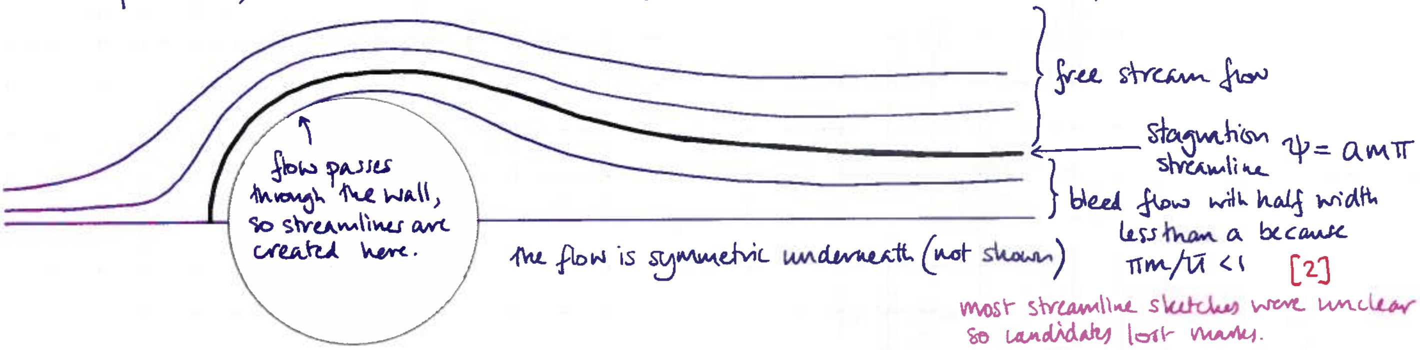
$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U \cos\theta \left(1 - \frac{a^2}{r^2} \right)$ At $r = a$, $u_r = 0$, as required. [2]

This was well answered. A common mistake was to omit the b.c. at $r \to \infty$.

(b) We need to add a flow element that creates an axisymmetric velocity field with radial outflow. A single source at the origin achieves this. The volumetric flowrate of this source is $2\pi a m$, where m is the volumetric flowrate per unit length at the perimeter of a cylinder of radius a. Far downstream the fluid has speed U so the width of the stream is $2\pi a m / U$. [4]   This was well answered

(c) $\psi = U \sin\theta \left(r - \frac{a^2}{r} \right) + \frac{2\pi a m}{2\pi}\theta = U \sin\theta \left(r - \frac{a^2}{r} \right) + a m \theta$ This was well answered

Far upstream, $r \to \infty$ and $\theta \to \pi$. Therefore $\psi \to U r \sin\theta + a m \pi = \cancel{Uy}^{\,0} + a m \pi = a m \pi$ [2]



[2]

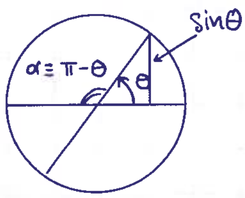
most streamline sketches were unclear so candidates lost marks.

(d) The stagnation streamline satisfies $\psi = U \sin\theta \left(r - \frac{a^2}{r} \right) + a m \theta = a m \pi$

Solve for $r(\theta)$ along this streamline: $r - \frac{a^2}{r} + \frac{a m}{U}\frac{(\theta - \pi)}{\sin\theta} = 0$

Define $\alpha \equiv \pi - \theta$

$\sin\alpha = \sin\theta$



$\Rightarrow r - \frac{a^2}{r} - 2\lambda = 0$ $2\lambda \equiv \frac{a m}{U}\frac{\alpha}{\sin\alpha}$

$\Rightarrow r^2 - 2\lambda r - a^2 = 0$

$\Rightarrow r = \lambda \pm \left(\lambda^2 + a^2 \right)^{1/2}$ Check: if $\lambda = 0$, $r = \pm a$ as required.

consider only +ve root, in top half. [4]

Only around 10% of candidates managed this.

(e) $\frac{r}{a} = \frac{\lambda}{a} + \left\{ 1 + \left(\frac{\lambda}{a} \right)^2 \right\}^{1/2}$ where $\frac{\lambda}{a} \equiv \frac{m}{2U}\frac{\alpha}{\sin\alpha}$ n.b. $\frac{\alpha}{\sin\alpha} \to 1$ as $\alpha \to 0$

The closest distance is at $\alpha = 0$, where $\frac{\lambda}{a} \to \frac{m}{2U}$, which is small.

By inspection, if $(\lambda/a)^2 \ll 1$ then $\frac{r}{a} \simeq 1 + \frac{\lambda}{a}$ $\Rightarrow$ closest distance $= r - a = \lambda = \frac{m a}{2U}$

To be rigorous, perform a Taylor expansion in λ/a around $\lambda/a = 0$:

$\frac{r}{a} \simeq \frac{\lambda}{a} + 1 + \frac{1}{2} 2 \left(\frac{\lambda}{a} \right)^2 + \ldots = 1 + \frac{\lambda}{a} + \left(\frac{\lambda}{a} \right)^2 + \ldots$ to give the same result. [4]

Very few completed this. A common error was to look for the minimum of the answer to (d), when it is clear that this minimum will occur at $\alpha = 0$.

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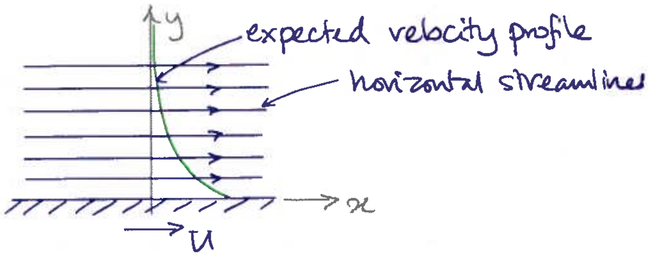
3.

(a) ① The fluid moves horizontally so $\underset{\sim}{u} = \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix}$

The flow is uniform in the x-direction so all
② partial derivatives with respect to x, z are zero.



③ The pressure gradient must be zero to avoid infinite pressure at $x = \pm$ infinity.

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underset{\sim}{u}$$ Three component Navier-Stokes

We need only to consider the u-component because v and w are zero from ①.

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \left(u \underbrace{\cancel{\frac{\partial}{\partial x}}}_{0 \leftarrow ②} + \underbrace{\cancel{v}}_{0 \leftarrow ①} \frac{\partial}{\partial y} + \underbrace{\cancel{w}}_{① \rightarrow 0} \frac{\partial}{\partial z} \right) u = -\frac{1}{\rho} \underbrace{\cancel{\frac{\partial p}{\partial x}}}_{0 \leftarrow ③} + \nu \left(\underbrace{\cancel{\frac{\partial^2}{\partial x^2}}}_{0 \leftarrow ②} + \frac{\partial^2}{\partial y^2} + \underbrace{\cancel{\frac{\partial^2}{\partial z^2}}}_{② \rightarrow 0} \right) u$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad [1]$$ as required.

Only simple arguments were required here. Some candidates repeated Prandtl's b'layer approximations, which were not appropriate. [4]

(b) $$\xi \equiv y (\nu t)^{-1/2} \quad \Rightarrow \quad \left.\frac{\partial \xi}{\partial y}\right|_t = (\nu t)^{-1/2} \; ; \; \left.\frac{\partial \xi}{\partial t}\right|_y = -\frac{1}{2} y (\nu t)^{-3/2} \nu = -\frac{1}{2} \frac{\xi}{t}$$

$$\left.\frac{\partial u}{\partial t}\right|_y = \left.\frac{\partial \xi}{\partial t}\right|_y \frac{du}{d\xi} = -\frac{1}{2} \frac{\xi}{t} u'$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_t = \left.\frac{\partial}{\partial y}\right|_t \left\{ \left.\frac{\partial \xi}{\partial y}\right|_t u' \right\} = \left.\frac{\partial}{\partial y}\right|_t \left\{ (\nu t)^{-1/2} u' \right\} = (\nu t)^{-1/2} \left.\frac{\partial \xi}{\partial y}\right|_t u'' = (\nu t)^{-1} u''$$

This was answered well by almost all candidates.

Substitute into [1] $\Rightarrow$ $$-\frac{1}{2} \frac{\xi}{t} u' = \nu (\nu t)^{-1} u''$$

$$\Rightarrow \quad 2u'' + \xi u' = 0 \qquad [2]$$ as required. [8]

(c) Boundary conditions are $u(y=0, t) = U$ and $u(y=\infty, t) = 0$ [3a]

$\Rightarrow$ $u(\xi = 0) = U$ and $u(\xi = \infty) = 0$ [3b]

Integrate [2] once by separating the variables u' and ξ:

$$[2] \rightarrow 2 \frac{du'}{d\xi} = -\xi u'$$

$$\Rightarrow \quad 2 \int \frac{du'}{u'} = -\int \xi \, d\xi$$

$$\Rightarrow \quad 2 \ln u' = -\frac{\xi^2}{2} + C_1$$

$$\Rightarrow \quad u' = C_2 \exp\left\{ -\frac{\xi^2}{4} \right\}$$

$$\rightarrow \quad \frac{du}{d\xi} = C_2 \exp\left\{ -\frac{\xi^2}{4} \right\}$$

$$\Rightarrow \quad \int du = C_2 \int \exp\left\{ -\frac{\xi^2}{4} \right\} d\xi$$

$$\Rightarrow \quad u = C_2 \sqrt{\pi} \left\{ \operatorname{erf}(\xi/2) + C_3 \right\}$$

$\operatorname{erf}(\xi/2) \rightarrow 1$ as $\xi \rightarrow \infty$ $\Rightarrow$ $C_3 = -1$ for [3b]

$\operatorname{erf}(0) = 0$ $\Rightarrow$ $U = -C_2 \sqrt{\pi}$ for [3a]

Now integrate again by separating the variables u and ξ

$$\Rightarrow u(y,t) = U\left\{1 - \operatorname{erf}\left(\tfrac{\xi}{2}\right)\right\} = U\left\{1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right\} \quad [4]$$

[4]

Few candidates managed this, even though it was a relatively simple integration. Many tried to substitute in a guess for the solution, but chose exponentials instead of the erf given. Around 1/3 candidates tried to use standard boundary layer boundary conditions: $u(0)=0$ and $u'(\infty)=0$, showing lack of adaptability to a new problem.

(d) The governing equations [1] and [2] are linear in u. Consequently if $u_1(y,t)$ is a solution and $u_2(y,t)$ is a solution then u_1+u_2 is also a solution.

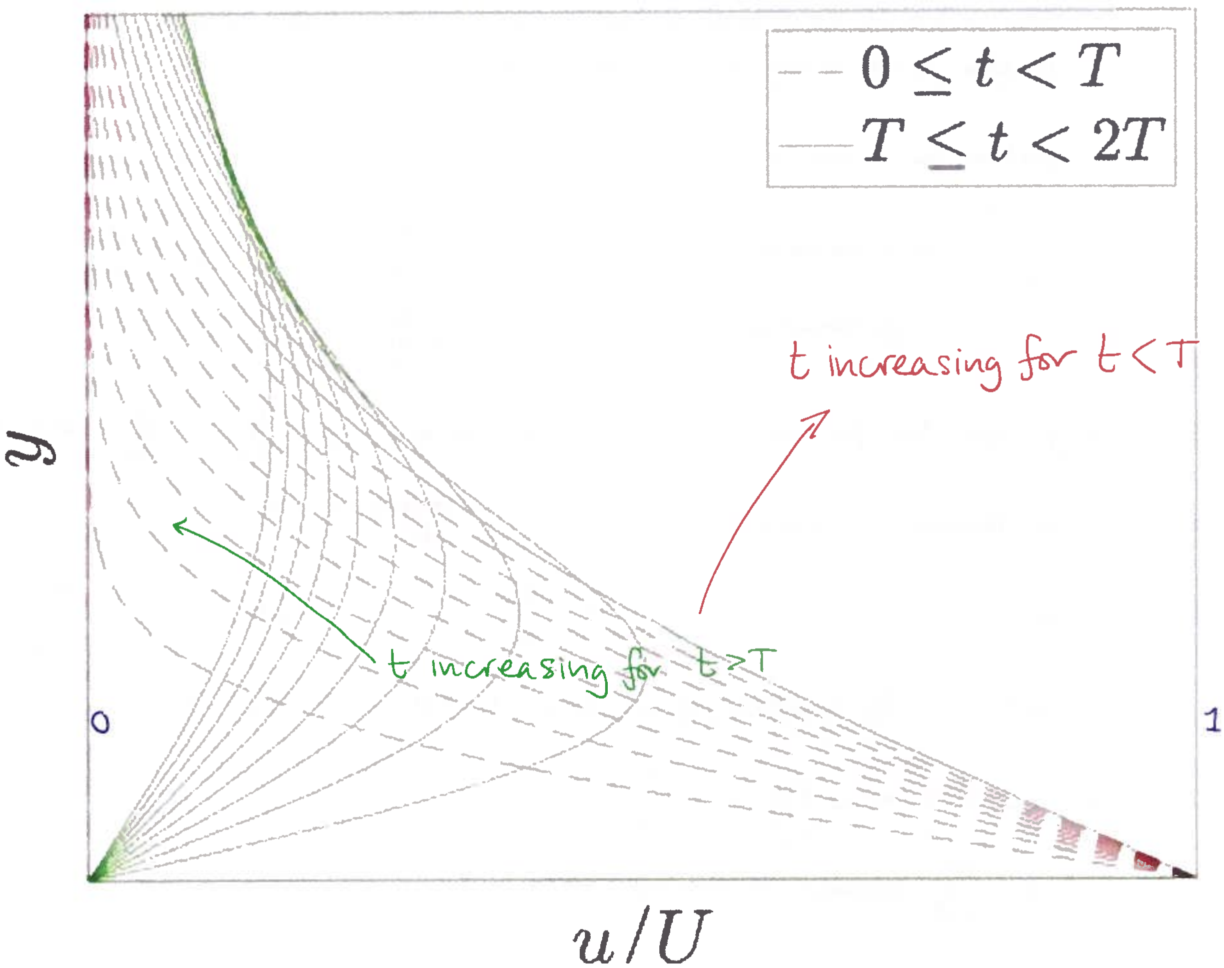
After time T, the flow can be modelled as the superposition of a flow with plate speed U that started at $t=0$ and a flow with plate speed $-U$ that started at T.

$$\Rightarrow u(x,t) = U\left\{1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right\} - U\left\{1 - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu(t-T)}}\right)\right\}$$

$$= U\left\{\operatorname{erf}\left(\frac{y}{2\sqrt{\nu(t-T)}}\right) - \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right\}$$

Very few candidates managed this.

For information, this is plotted below



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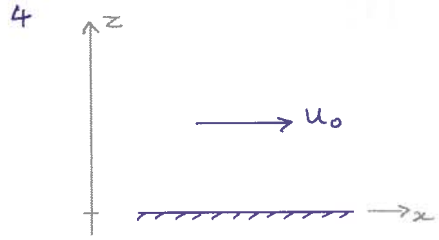
4



$$2\underset{\sim}{\Omega}\times\underset{\sim}{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{u} \qquad \underset{\sim}{\Omega} = \begin{bmatrix}0\\0\\\Omega\end{bmatrix} \qquad \underset{\sim}{u} = \begin{bmatrix}u_0\\0\\0\end{bmatrix} \text{ outside b'layer}$$

(a) Scaling analysis: $|\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}| \sim U^2/L$; $|\underset{\sim}{\Omega}\times\underset{\sim}{u}| \sim \Omega U$

If the inertial term is negligible then $U^2/L \ll \Omega U \Rightarrow U/L\Omega \ll 1$ — known as "Rossby number"

Most students answered this well [2]

(b) In the rotating frame of reference, a pressure gradient is required to maintain a uniform flow. In the free stream, the flow is inviscid, so:

$$2\underset{\sim}{\Omega}\times\underset{\sim}{u} = -\frac{1}{\rho}\nabla p$$

$$\Rightarrow \nabla p = -2\rho\begin{bmatrix}0\\0\\\Omega\end{bmatrix}\times\begin{bmatrix}u_0\\0\\0\end{bmatrix} = -2\rho\begin{bmatrix}0\\u_0\Omega\\0\end{bmatrix} \Rightarrow \frac{\partial p}{\partial x} = 0,\ \frac{\partial p}{\partial y} = -2\rho u_0\Omega,\ \frac{\partial p}{\partial z} = 0$$

Most students answered this well [2]

(c) $\nabla\cdot\underset{\sim}{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ but u and v are functions of z only so $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

$\Rightarrow \frac{\partial w}{\partial z} = 0$. Now, $w = 0$ at $z = 0$ because fluid cannot pass through the boundary. Therefore $w = 0$ everywhere. Most students answered this well [2]

$\nabla^2 = \frac{\partial^2}{\partial z^2}$

(d) The momentum equations for u and v are found from $2\underset{\sim}{\Omega}\times\underset{\sim}{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{u}$

$\underset{\sim}{\Omega}\times\underset{\sim}{u} = \begin{bmatrix}0\\0\\\Omega\end{bmatrix}\times\begin{bmatrix}u\\v\\w\end{bmatrix} = \begin{bmatrix}-v\Omega\\u\Omega\\0\end{bmatrix}$ so x-momentum eq. is: $-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$ Most students answered this well

y-momentum eq. is: $2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\quad}{\partial z^2}$ [2]

(e) In the boundary layer, the pressure takes the free stream value $\Rightarrow \frac{\partial p}{\partial x} = 0,\ \frac{\partial p}{\partial y} = -2\rho u_0\Omega$

$\Rightarrow$ x-momentum eq. becomes: $-2\Omega v = \nu\frac{\partial^2 u}{\partial z^2}$ Most students answered this well (1a)

y-momentum eq. becomes: $2\Omega u = 2\Omega u_0 + \nu\frac{\partial^2 v}{\partial z^2} \Rightarrow 2\Omega(u - u_0) = \nu\frac{\partial^2 v}{\partial z^2}$ (1b)

[4]

(f) Define $f = u - u_0 + iv$. By inspection, if we take Eq.(1b) $- i\times$ Eq.(1a) we will obtain an eq. for f:

$$\Rightarrow 2\Omega(u - u_0) + i2\Omega v = \nu\frac{\partial^2 v}{\partial z^2} - i\nu\frac{\partial^2 u}{\partial z^2}$$

$$\Rightarrow 2\Omega f = -i\nu\frac{\partial^2 f}{\partial z^2} \text{ because } u_0 \text{ does not depend on } z.$$

Now, by construction in part (c), u and v depend only on z. Therefore f depends only on z and $\partial^2 f/\partial z^2$ collapses to the ordinary derivative $d^2 f/dz^2$.

$$\Rightarrow 2\Omega f + i\nu f'' = 0$$

Try a solution of the form $f = Fe^{\lambda z}$ $\Rightarrow f'' = \lambda^2 f$

$$2\Omega f + i\nu f'' = 0$$

$$\Rightarrow 2\Omega + i\nu\lambda^2 = 0 \quad \Rightarrow \lambda^2 = \left(\frac{2\Omega}{\nu} i\right)^{1/2} \quad \Rightarrow \lambda = \pm(1+i)(\Omega/\nu)^{1/2}$$

The general solution is therefore:

$$f = A\exp\left\{-(1+i)(\Omega/\nu)^{1/2} z\right\} + B\exp\left\{+(1+i)(\Omega/\nu)^{1/2} z\right\}$$

Many students managed this.

[6]

(g) Note that A and B can be complex and that f can oscillate in z.

At $z=\infty$, u and v are bounded ($u = u_0$ and $v = 0$) but the second term above is not, so $B = 0$.

At $z = \infty$, $f = 0$

At $z = 0$, $u = 0$ and $v = 0$ $\Rightarrow f = -u_0$

So the general solution is $f = -u_0 \exp\left\{-(1+i)(\Omega/\nu)^{1/2} z\right\}$

Only a few students did this

$$\Rightarrow u = u_0 + \mathrm{Real}\{f\} = u_0\left[1 - \cos(\Omega/\nu)^{1/2} z \, \exp\left\{-(\Omega/\nu)^{1/2} z\right\}\right]$$

$$\text{and} \quad v = \mathrm{Imag}\{f\} = u_0 \sin(\Omega/\nu)^{1/2} z \, \exp\left\{-(\Omega/\nu)^{1/2} z\right\}$$

[2]

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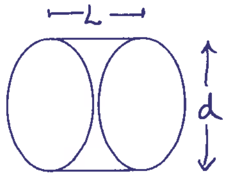
5.
(a)



Consider a control volume with length L.

The force due to the pressure gradient is $\frac{\pi d^2}{4}\frac{dp}{dx}L$

This is balanced by the force due to the walls $= \tau_w \pi d L$

$$\frac{dp}{dx} = \frac{4\tau_w}{d}$$

$$\Rightarrow f = \frac{dp/dx}{\frac{1}{2}\rho V^2/d} = \frac{4\tau_w}{\frac{1}{2}\rho V^2} = \frac{8\tau_w}{\rho V^2}$$

Remarkably few candidates managed this. A common mistake was to overcomplicate the question.

[2]

(b) (N.b. we ignore the viscous sublayer (note that $u(r)$ is singular there) but this makes only a small difference to the integral performed in this question because $\ln y$ tends to infinity very slowly as $y \to 0$.)

The mean velocity, V, and the friction velocity, u^*, can be related by considering the flowrate, Q:

$$Q = \int_{r=0}^{R} 2\pi r u \, dr = \pi R^2 V$$

Define $y = R - r$ such that $Q = \int_{y=0}^{y=R} 2\pi u^*(R-y)\left\{\frac{1}{\kappa}\ln\left(\frac{u^* y}{\nu}\right) + B\right\} dy$

$$\Rightarrow Q = 2\pi u^*\left\{R^2 B - \frac{R^2 B}{2} + \underbrace{\int_0^R \frac{R}{\kappa}\ln\left(\frac{u^* y}{\nu}\right) dy}_{①} - \underbrace{\int_0^R \frac{1}{\kappa} y \ln\left(\frac{u^* y}{\nu}\right) dy}_{②}\right\}$$

For ① we use $\int \ln x \, dx = x \ln x - x$ with $x = \frac{u^* y}{\nu}$ $\Rightarrow \frac{dx}{dy} = \frac{u^*}{\nu}$

$$\Rightarrow ① = \frac{R}{\kappa}\int_0^{u^*R/\nu} \frac{\nu}{u^*}\ln x \, dx = \frac{R\nu}{\kappa u^*}\Big[x \ln x - x\Big]_0^{u^*R/\nu} = \frac{R\nu}{\kappa u^*}\left\{\frac{u^*R}{\nu}\ln\left(\frac{u^*R}{\nu}\right) - \frac{u^*R}{\nu}\right\} = \frac{R^2}{\kappa}\left\{\ln\left(\frac{u^*R}{\nu}\right) - 1\right\}$$

For ② we use $\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$ with $x = \frac{u^* y}{\nu}$ as above.

$$\Rightarrow ② = \frac{1}{\kappa}\int_0^{u^*R/\nu}\left(\frac{\nu}{u^*}\right)^2 x \ln x \, dx = \frac{1}{\kappa}\left(\frac{\nu}{u^*}\right)^2\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_0^{u^*R/\nu} = \frac{1}{\kappa}\left(\frac{\nu}{u^*}\right)^2\left\{\frac{1}{2}\left(\frac{u^*R}{\nu}\right)^2\ln\left(\frac{u^*R}{\nu}\right) - \frac{1}{4}\left(\frac{u^*R}{\nu}\right)^2\right\}$$

$$= \frac{1}{\kappa}\left\{\frac{R^2}{2}\ln\left(\frac{u^*R}{\nu}\right) - \frac{R^2}{4}\right\} = \frac{R^2}{\kappa}\left\{\frac{1}{2}\ln\left(\frac{u^*R}{\nu}\right) - \frac{1}{4}\right\}$$

$$\Rightarrow Q = 2\pi u^*\left\{\frac{R^2 B}{2} + \frac{R^2}{\kappa}\left\{\ln\left(\frac{u^*R}{\nu}\right) - 1\right\} - \frac{R^2}{\kappa}\left\{\frac{1}{2}\ln\left(\frac{u^*R}{\nu}\right) - \frac{1}{4}\right\}\right\}$$

$$= \pi R^2 u^*\left\{B + \frac{1}{\kappa}\left\{\ln\left(\frac{u^*R}{\nu}\right) - \frac{3}{2}\right\}\right\} = \pi R^2 V$$

$$\Rightarrow V = u^*\left\{B + \frac{1}{\kappa}\left\{\ln\left(\frac{u^*R}{\nu}\right) - \frac{3}{2}\right\}\right\}$$

Only a few candidates managed this.

[8]

(c) $u^* = (\tau_w/\rho)^{1/2} \Rightarrow \tau_w = \rho u^{*2}$; $f = \dfrac{8\tau_w}{\rho V^2} = 8\left(\dfrac{u^*}{V}\right)^2 \Rightarrow \dfrac{u^*}{V} = \left(\dfrac{f}{8}\right)^{1/2}$

From (b), $V = u^*\left\{B + \dfrac{1}{\kappa}\left\{\ln\left(\dfrac{u^*R}{\nu}\right) - \dfrac{3}{2}\right\}\right\}$ $\quad \dfrac{u^*R}{\nu} = \dfrac{u^*}{V}\dfrac{1}{2}\dfrac{Vd}{\nu} = \dfrac{1}{2}\dfrac{u^*}{V}Re = \dfrac{1}{2}Re\left(\dfrac{f}{8}\right)^{1/2}$

$$\Rightarrow \left(\frac{8}{f}\right)^{1/2} = B + \frac{1}{\kappa}\left\{\ln\left(\frac{1}{2}Re\left(\frac{f}{8}\right)^{1/2}\right) - \frac{3}{2}\right\}$$

This is an implicit relationship between Re and f.

only a few managed this [4]

(d) max velocity is at $r=0$ $\Rightarrow \dfrac{u_{max}}{u^*} = B + \dfrac{1}{\kappa}\ln\left(\dfrac{u^*R}{\nu}\right)$

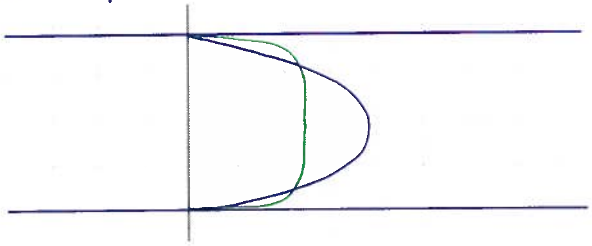
From (b), $\dfrac{V}{u^*} = B + \dfrac{1}{\kappa}\ln\left(\dfrac{u^*R}{\nu}\right) - \dfrac{3}{2\kappa}$

$\Rightarrow \dfrac{u_{max}}{u^*} = \dfrac{V}{u^*} + \dfrac{3}{2\kappa} = \dfrac{V}{u^*}\left(1 + \dfrac{3}{2\kappa}\dfrac{u^*}{V}\right)$

$$\Rightarrow \frac{u_{max}}{V} = 1 + \frac{3}{2\kappa}\frac{u^*}{V} = 1 + \frac{3}{2\kappa}\left(\frac{f}{8}\right)^{1/2} = 1 + \frac{3}{4\kappa}\left(\frac{f}{2}\right)^{1/2} \quad \text{from (c)}$$

[4]

(e) The turbulent velocity profile is flat in the centre and drops off sharply near the wall:
The laminar profile is parabolic.



This was well answered by most candidates. [2]

J. Li & M. Juniper 2024

An aerofoil has the following camberline

$$\frac{y}{c} = \frac{1}{5}\left[\frac{x}{c} - \left(\frac{x}{c}\right)^2\right] \qquad 0 \le \frac{x}{c} < 0.5$$

$$\frac{y}{c} = \frac{3}{20}\left[1 - \frac{x}{c}\right] \qquad 0.5 \le \frac{x}{c} \le 1$$

a) Use thin aerofoil theory to determine the zero-lift angle of attack.

b) At what angle of attack (in degrees) would you expect the max adverse pressure gradient along the airfoil surface to be a minimum? Explain your answer with sketches.

To determine zero lift α, we need first two coefficients in Fourier series for camberline

$$-2\frac{dy_c}{dx} = g_0 + \sum g_n \cos n\theta$$

$$\frac{dy_c}{dx} = \frac{1}{5}\left(1 - 2\frac{x}{c}\right) \quad \text{for } 0 \le x \le \frac{c}{2}$$

and $$\frac{dy_c}{dx} = -\frac{1}{10} \quad \text{for } \frac{c}{2} < x \le c$$

Using usual trafo: $\frac{x}{c} = \frac{1}{2}(1+\cos\theta)$

so: $$-2\frac{dy}{dx} = \frac{2}{5}\cos\theta \quad \text{for } \frac{\pi}{2} \le \theta \le \pi$$

$$-2\frac{dy}{dx} = \frac{1}{5} \quad \text{for } 0 \le \theta < \frac{\pi}{2}$$

$$g_0 = \frac{1}{\pi}\int_0^{\pi}\left(-2\frac{dy}{dx}\right)d\theta = \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}}\frac{d\theta}{5} + \int_{\frac{\pi}{2}}^{\pi}\frac{2}{5}\cos\theta\, d\theta\right] = \frac{1}{\pi}\left(\frac{\pi}{10} - \frac{2}{5}\right) = -0.027$$

$$g_1 = \frac{2}{\pi}\int_0^{\pi}\left(-2\frac{dy}{dx}\right)\cos\theta\, d\theta = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{5}d\theta + \int_{\frac{\pi}{2}}^{\pi}\frac{2}{5}\underbrace{\cos^2\theta}_{=\frac{1+\cos 2\theta}{2}}\, d\theta\right]$$

$$g_1 = \frac{2}{5\pi} + \frac{1}{5} = 0.327$$

$$\alpha_0 = -\left(\frac{g_0}{2} + \frac{g_1}{4}\right) = -0.0683 \qquad (-3.9^0)$$

b) This is design angle – see notes

$$\alpha_d = -\frac{g_0}{2} = 0.0135 \qquad (0.782^0)$$

At this angle the suction peak at LE disappears – sketches in notes for $n \neq 1$. The integrals in (70) are straightforward, giving

$$c_l = 2\pi\alpha + \pi\left[g_0 + \frac{g_1}{2}\right]. \tag{72}$$

Notes:

(i) Unsurprisingly, because the linear solution allows us to separate the incidence and camber effects, the lift-curve slope is that of the flat plate (at small angles): 2π.

(ii) The camber contribution comes from just the first two terms in the Fourier series for $-2(dy_c/dx)$.

(iii) The lift coefficient of the cambered aerofoil can alternatively be expressed as

$$c_l = 2\pi(\alpha - \alpha_0), \tag{73}$$

where

$$\alpha_0 = -\left(\frac{g_0}{2} + \frac{g_1}{4}\right) \tag{74}$$

is the angle at which the aerofoil produces zero lift. For the positive-camber case, where $g_0 + g_1/2 > 0$, it will be negative.

The effect of camber on the lift curve is shown schematically in Figure 16.

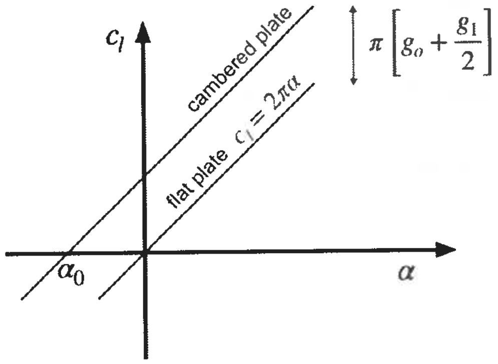


Figure 16: Thin aerofoil theory lift predictions for flat plate and cambered aerofoils.

The aerofoil pitching moment (anti-clockwise positive, about the leading edge) is given by the integral of elemental lift contributions multiplied by their moment arms, i.e.

$$M = \int_0^c (p_l - p_u) l\,dl. \tag{75}$$

The pitching moment coefficient, c_m, is defined as the non-dimensionalisation of M by $\frac{1}{2}\rho U^2 c^2$, and (from (67)) $p_l - p_u = -\rho U\gamma$, so

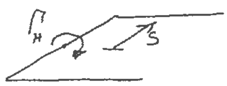
$$c_m = \frac{2}{c^2}\int_0^c -\frac{\gamma}{U} l\,dl. \tag{76}$$

Substituting (69) for γ and using $l = (c/2)(1 + \cos\phi)$ gives

An aircraft (weight W) is in steady flight (air density rho) with a velocity V. It features a wing with semi-span s and an aspect ratio AR.
a) Determine the induced drag coefficient from lifting line theory as a function of rho, W, V, s and AR. You may assume elliptical wing loading.
b) Using the simple horseshoe vortex model, estimate the induced drag coefficient. Explain why it is different from the results from a)
c) Show that a modified horseshoe vortex model which makes use of the effective span s' and an effective downwash w' can provide an improved estimate of the induced drag coefficient. Express s' and w' as functions of m, V, AR and rho.
d) compare the relationship between induced drag and lift coefficients for the lifting line model with that from the modified horseshoe vortex model used in c).
e) what aspects of the wing flow are better described by the modified horseshoe vortex model and which weaknesses remain?

$$L = W = C_L \tfrac{1}{2}\rho V^2 A \qquad C_L = \frac{2W}{\rho V^2 A} \qquad A_R = \frac{4s^2}{A}$$

a) $$C_{D_{i,LL}} = \frac{C_L^2}{\pi A_R} = \frac{4W^2}{\rho^2 V^4 A^2 \pi A_R} = \frac{W^2 A_R}{4\pi \rho^2 V^4 s^4}$$

b) 

$$L = \rho V \Gamma_H 2s = W \quad \therefore \Gamma_H = \frac{W}{2\rho V s}$$

Downwash at centre: $w_H = \frac{\Gamma}{2\pi s}$ (see notes)

Assume w_H is uniform across wing:

$$D_i = \rho\, w_H \Gamma_H \cdot 2s = \frac{\rho}{\pi}\Gamma_H^2 = \frac{W^2}{4\pi\rho V^2 s^2}$$

$$C_{D_{i,H}} = \frac{D_i}{\frac{1}{2}\rho V^2 A} = \frac{W^2}{2\pi\rho^2 V^4 s^2 A} = \frac{W^2 A_R}{8\pi\rho^2 V^4 s^4} = \frac{1}{2} C_{D_{i,LL}}$$

The horseshoe vortex model underpredicts induced drag.
See notes (section 3.4)

c) now, use improved span so that $\Gamma_H = \Gamma_{LL}(y=0) = \Gamma_0$ (same lift)

$$\Gamma_0 = \frac{2W}{\pi\rho V s} \text{ (elliptic loading - data sheet)} = \frac{W}{2\rho V s'} \text{ (horseshoe model, see above)}$$

$$s' = \frac{\pi}{4} s \quad \text{(smaller than } s\text{)}$$

also assume
$$w' = \frac{\Gamma_H}{4s} \quad \text{(lifting line for elliptic loading, use geometric } s\text{)}$$

$$D_i = \rho w' \Gamma_H 2s' = \frac{\pi}{8}\rho \Gamma_H^2$$

$$C_{D_i} = \frac{D_i}{\frac{1}{2}\rho V^2 A} = \frac{2\pi\rho\Gamma_H^2}{8\rho V^2 A} = \frac{\pi W^2}{4 \cdot 4\rho^2 V^2 s'^2 V^2 A}$$

$$= \frac{W^2}{\pi\rho^2 V^4 s^2 A} = \frac{W^2 A_R}{4\pi\rho^2 V^4 s^4} \quad \text{same result as a)}$$

d) $$C_L = \frac{W}{\frac{1}{2}\rho V^2 A} \qquad W = C_L \cdot \tfrac{1}{2}\rho V^2 A$$

$$\text{so } C_{D_i} = \frac{C_L^2 \cancel{\frac{1}{4}\rho^2 V^4} A^{\cancel{2}}}{\cancel{\rho^2}\pi \cancel{V^4} s^2 A} = C_L^2 \frac{A}{\pi 4 s^2} = \underline{\underline{\frac{C_L^2}{\pi A_R}}}$$

This is the lifting line result.

e) see notes, section 3.4. induced drag and lift/drag behaviour are correct but flow near tips is wrong.

3.3.6 Viscous drag estimation

If one has drag data for the section shape(s) used in the wing, then the results of the lifting-line calculation can be used to provide an estimate of the viscous drag, via integration of the local section drag contributions. Recall that the local lift, $l(y)$, is equal to $\rho U \Gamma(y)$, so

$$\Gamma(y) = \frac{l(y)}{\rho U} = \frac{1}{2} U c(y) c_l(y). \tag{155}$$

This means that the local lift coefficient, c_l, can be calculated from the circulation distribution. The viscous drag coefficient, c_d, is typically given in 'polar' form as a function of c_l, so we now know the local drag, $\frac{1}{2}\rho U^2 c(y) c_d(y)$, and can thus calculate the overall drag coefficient via a numerical evaluation of the integral

$$C_{Dv} = \frac{1}{S} \int_{-s}^{s} c(y) c_d(y) dy. \tag{156}$$

3.4 Refinements to the horseshoe-vortex model

In light of the discussion of lifting line theory it is possible to make some adjustments to the horseshoe-vortex model to improve its accuracy. In the model, downwash was calculated at the centre and assumed constant along the wing. We now know this not to be the case unless the wing loading is elliptic. However, given that realistic wing shapes have lift distributions that are not too different from elliptical this simplification is not too bad in principle. Nevertheless, the downwash magnitude predicted by the horseshoe-vortex model is also wrong, for two reasons:
1) The actual downwash velocity is induced by a vortex sheet and not by two discrete vortices.
2) The tip vortex formation is more complex than described by the model, and the eventual vortex spacing is smaller than a wingspan.

We could address 1) by using the equation for elliptic downwash instead ($w_d = \Gamma_h/4s$). A further improvement of the horseshoe vortex model is to modify the spacing of the tip vortices downstream of the wing. To do so, we can consider the actual development of the wake, as shown schematically in Figure 52.

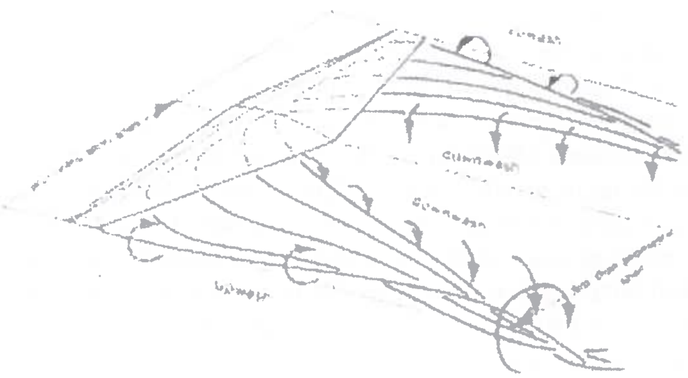


Figure 52: Formation of tip vortices

As a starting point, recall that vortex lines move with the flow (in an inviscid fluid). Now add the requirement that the flow is steady, so our thin, vortex-sheet, wake is fixed in space.

The only way these two statements can both be true is if the local flow velocity in the wake lies in the plane of the sheet. Thus, to a first approximation, we might expect the wake to consist of a flat vortex sheet more-or-less in the free-stream direction. In fact, though, the additional velocity components (associated with the bound and wake vorticity) result in the wake 'rolling up' into a pair of concentrated streamwise vortices (Figure 52). These eventually become the 'tip vortices' of the horseshoe-vortex model, however, as we can see they form somewhat inside the actual wing span, at a spacing called the *effective span* s'.

3.4.1 The effective span

The lift on the bound segment of the horseshoe vortex is $\rho U\Gamma.2s'$, and this tells us that the effective span must be less than the true span if we are to predict the correct lift, because the real bound circulation drops below Γ_0 as the wing tips are approached. For the particular case of the elliptic lift distribution, we have (from (140))

$$L = \rho U\Gamma_0.\frac{\pi s}{2}, \tag{157}$$

so

$$s' = \frac{\pi}{4}s. \tag{158}$$

In general, the requirement that the lift is correct says that $\rho U\Gamma_0.2s'$ must be equal to the integral of $\rho U\Gamma(y)$, i.e.

$$s' = \frac{1}{2\Gamma_0}\int_{-s}^{s}\Gamma(y)dy. \tag{159}$$

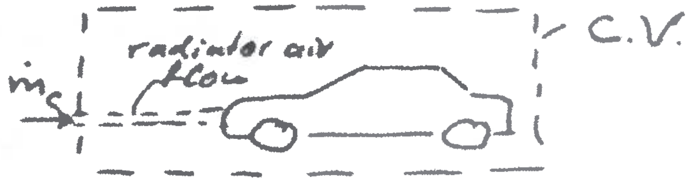
One can also ask whether the effective semi-span required to match the lift gives an accurate representation of the trailing vortex locations. The dynamics of wake roll-up are beyond the scope of this course, but it can be shown that the rolled-up vortices should end up at a spanwise coordinate equal to the 'centroid' of the shed vorticity distribution, i.e.

$$\bar{y} = \frac{1}{\Gamma_0}\int_{0}^{s}\left(-\frac{d\Gamma}{dy}\right)ydy. \tag{160}$$

Reassuringly, this turns out to be exactly the same as s', as given by (159). The proof of this result is left to the energetic reader (hint: integrate by parts).

Whether it is worth 'improving' the simple horseshoe-vortex model is debatable. Even with the above 'fixes' it can not give very accurate answers. The main value of the model lies in its ability to describe the main effects of other lifting surfaces and vortices in the vicinity (for example in formation flying or ground effect). Most of these effects are relatively well captured when using the simple model (where $\Gamma = \Gamma_0$ and $s' = s$) and the extra effort seems hardly worth it. However, please note that some of the examples paper answers (and old exam questions) are occasionally calculated with a more refined model using the above effective span or the actual downwash predicted by the lifting line model for elliptic wing loading (although I have made efforts to edit any inconsistencies).

a) See lecture notes



SFME: $F = \dot m \Delta V$ assume $V_{out} \approx 0$

so $D_{cooling} = \dot m_c \cdot (V_\infty - 0)$

and $c_{D_c} = \dfrac{\dot m_c V_\infty}{\frac{1}{2}\rho V_\infty^2 A} = \dfrac{2\dot m_c}{\rho V_\infty A}$

However, at motorway speeds the flow through the radiator is greater than on the stationary test bed.

$V_c = \dfrac{V_\infty}{5}$ (given)

so $\dot m_c = \dfrac{V_\infty}{5} \cdot \rho A_{c,eff}$

gives $\Delta c_{D_c} = \dfrac{2}{5}\dfrac{A_{c,eff}}{A} = 0.016$

Typical car $c_D \approx 0.3$, so this is 5% of total

b) Now, assume radiator is closed down (see notes) to only allow minimum required mass flow to engine. Then $\dot m_c = \rho \cdot 0.05 \frac{m^3}{s}$

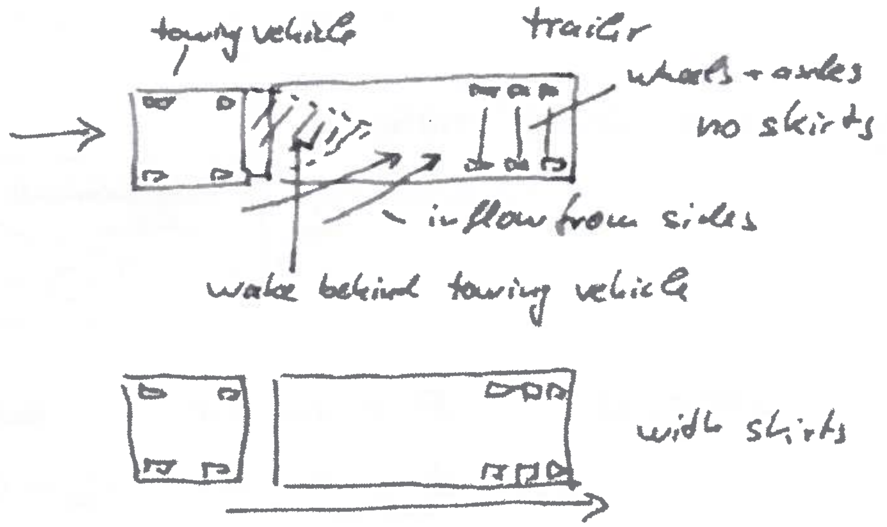
Assume $V_\infty = 31 \frac{m}{s}$ (70 mph)

from above: $c_{D_c} = \dfrac{2 \cdot 0.05}{31 \cdot 2.5} = 0.0013$

This is 8% of result from a), reduction of 92%

Achievable through 'active' radiators, e.g. using louvres.

c) See notes



Good answers mention:

- 30% of drag from underbody - wheels/axles/etc
- side skirts prevent air being entrained from side → see sketch
- this problem is worse in cross-wind
- entrained air has high momentum and hits drag-producing objects in rear (wheels etc.)

d) Many students confused fins with spoilers for downforce.

See notes. Good answers mention:

- record vehicles are highly streamlined
- from above, this makes them look (and behave) like aerofoils
- In crosswind, this generates sideforce
- Point of action ($\frac{1}{4}$ chord) usually ahead of Cg
- Can produce de-stabilising moment.
- Fin moves effective centre of lift behind Cg to stabilise vehicle