EGT2
ENGINEERING TRIPOS PART IIA

Monday 28 April 2025   9.30 to 12.40

Module 3A1

FLUID MECHANICS I

*Answer not more than **five** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book
Attachments:

- Incompressible Flow Data Card (2 pages);
- Boundary Layer Theory Data Card (1 page);
- 3A1 Data Sheet for Applications to External Flows (2 pages).

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 The snapping of a shrimp's claws can be modelled as the 2D inviscid flow between two flat plates, each of length R, hinged at the origin and subtending angle 2α, as shown in Fig. 1. The claws move together symmetrically, each with angular speed $\Omega = \pm\dot{\alpha}$, as shown in the figure.

(a) What is the kinematic boundary condition for $\mathbf{u} = (u_r, u_\theta)$ on the top plate? [10%]

(b) Show that the streamfunction $\psi = Axy$ satisfies this kinematic boundary condition on the top and bottom plates and find A in terms of Ω and α. Comment on how A varies with α. [20%]

(c) Calculate the pressure on the top and bottom plates and comment on the viscous effects that have been neglected in this model. [20%]

(d) Derive an expression for the horizontal momentum flowrate of the flow exiting the claws and comment on this mechanism for generating a fluid jet. [50%]

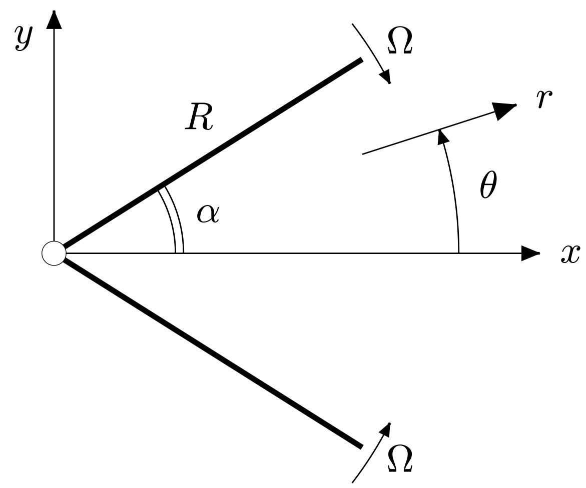

Fig. 1

2 Two co-rotating vortices of equal magnitude Γ are separated by a distance $2a$.

(a) Calculate the angular velocity, Ω, at which the vortices rotate around their mid-point. [10%]

(b) Confirm that $\psi = \Omega r^2/2$ is the streamfunction for clockwise solid body rotation at angular speed Ω. Show that the vorticity is non-zero and explain why a streamfunction can be defined for this flow, even though it is not irrotational. [20%]

(c) The x-axis is defined as the line that connects the two vortices. Write down the flow's streamfunction in this plane in terms of the distance to the origin, r, and the distances to the vortices, r_1 and r_2. [20%]

(d) Sketch

(i) the y-velocity as a function of position along the x-axis;

(ii) the streamlines for $|r| \gg a$;

(iii) the streamlines for $0 < |r| < 2a$;

(iv) the entire flow.

[50%]

3 The semicircular open channel shown in Fig. 2 has outer radius R, inner radius $3R/4$, and a uniform rectangular cross-section. It contains water with height, $h(r)$, where $h \ll R$, which arrives with uniform velocity from a straight channel with the same cross-section.

(a) Assuming that the flow remains inviscid and irrotational, derive an expression for the velocity, $\mathbf{u}(r)$, and the water height, $h(r)$, in the semicircular region of the channel. Explain your reasoning. [20%]

(b) Consider a line of fluid between $3R/4 \leq r \leq R$ that starts at the entrance to the semicircular region and takes time T to convect around the inner radius. Assuming that the flow is inviscid and irrotational, carefully sketch the location of this line at four times: $T/4, T/2, 3T/4, T$. [30%]

(c) In reality, a boundary layer has developed before the entrance to the semicircular channel. The initial vorticity, ω_0, is uniform along the line at the entrance in part (b). Along this vortex line, the local vorticity divided by the local length remains constant. Calculate the initial rate of growth of its vorticity, $\mathrm{d}\omega/\mathrm{d}t$, as a function of r. [30%]

(d) What secondary flow does this vorticity create in the semicircular channel. Sketch the streamlines of this flow in the (r, z) plane. [10%]

(e) How does the pressure, $p(r)$, at the bottom of the channel affect the boundary layer flow? Is this consistent with your answer to part (d) of this question? [10%]

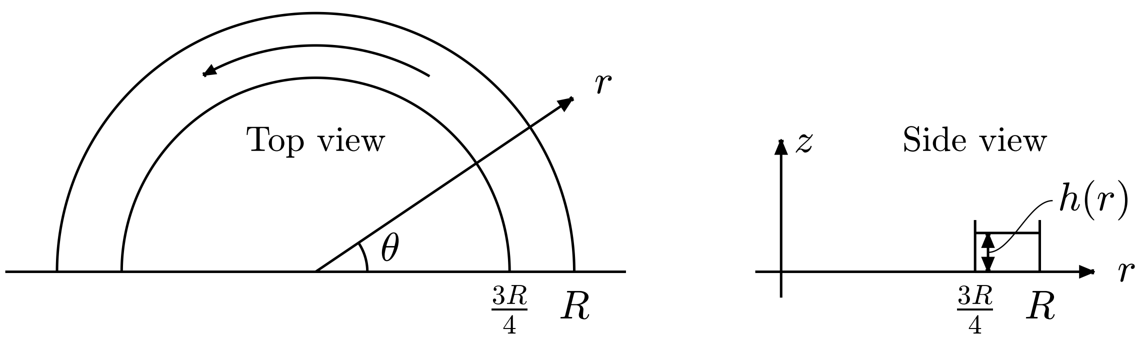

Fig. 2

4 A laminar boundary layer on a flat plate has a velocity profile

$$\frac{U}{U_1} = \sin\left(\frac{\pi\eta}{2}\right) \quad \text{for} \quad 0 \leq \eta \leq 1$$

where $\eta = y/\delta$ and U_1 is the free-stream velocity. The length δ is a measure of the boundary layer thickness, which is zero at the leading edge of the plate.

(a) By considering the boundary layer equation very close to the wall, show that the pressure gradient must be zero. [20%]

(b) Find:

(i) The displacement thickness δ^*;

(ii) The momentum thickness θ;

(iii) The shape factor H;

(iv) The local skin friction coefficient C_f'. [40%]

(c) Write down the momentum integral equation in terms of the quantities in part (b) and find an expression for $\delta(x)$. [20%]

(d) Find the variation of C_f' with $Re_x = U_1 x/\nu$, where x is the distance from the leading edge of the plate and ν is the kinematic viscosity. [20%]

5 Thwaites defined a dimensionless parameter at the wall of a laminar boundary layer:

$$m = \frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_w$$

where $U = U(x)$ is the freestream velocity and θ is the boundary layer's momentum thickness.

(a) (i) Show that

$$m = -\frac{\theta^2}{\nu}\frac{dU}{dx}$$

where ν is the fluid's kinematic viscosity. [20%]

(ii) From the momentum integral equation, Thwaites deduced that

$$U\frac{d(\theta^2)}{dx} = \nu(0.45 + 6m)$$

Deduce a formula for θ^2 as a function of the location of interest in terms of the freestream velocity U. You may assume that either $U(a) = 0$ or $\theta(a) = 0$ at one location $x = a$. [30%]

(b) Consider the flow through a divergent channel whose walls are straight.

(i) The wall is assumed to begin at $x = a$, where the entrance velocity of the freestream is U_0. The freestream flow is given by

$$U(x) = U_0\frac{a}{x}$$

where U_0 is a constant. Use the formulae from part (a) to calculate θ and m, assuming that $\theta(a) = 0$. [25%]

(ii) Assuming that the separation point is given by $m = 0.09$, find its location and the pressure coefficient there, using the pressure and velocity at $x = a$ as the reference quantities. Comment on your results. [25%]

6 An aircraft has a rectangular wing with semispan 5 m and aspect ratio 2π. It is cruising at 50 m s^{-1} in air with density 1.2 kg m^{-3}. The wing has an elliptic lift distribution and a wing loading of 450 N m^{-2}.

(a) What is the geometric angle of attack at the wing tip in degrees? [35%]

(b) What is the geometric angle of attack at the root in degrees? [40%]

(c) The flight speed is reduced at the same altitude. Explain why it is not possible to maintain an elliptic lift distribution without changing the wing geometry. Sketch how the resulting lift distribution compares to the lift distribution at cruise. [25%]

7 (a) Sketch the distribution of pressure coefficient, c_p, on a typical aerofoil and indicate regions where flow separation might be expected. [20%]

(b) With the aid of simple sketches, describe the development of (i) trailing edge stall and (ii) 'bubble-burst' stall with increasing wing loading. [30%]

(c) Sketch the lift coefficient, c_l, as a function of angle of attack, α, for trailing edge stall and 'bubble-burst' stall. State, with reasoning, which type of stall would be most appropriate for a trainer aircraft used to teach new pilots. [10%]

(d) Describe and explain the design changes that can delay the onset of trailing edge stall and 'bubble-burst' stall. [15%]

(e) A small-scale exact geometric model of a full-scale aircraft often has different stall behaviour from the full-scale aircraft. Explain this observation and describe how the behaviour of the small-scale model can be brought closer to that of the full-scale aircraft. [25%]

8 A very simple hatchback car with straight sides is to be improved aerodynamically.

(a) The underbody is fitted with a short diffuser at the rear. The diffuser angle, α, is measured relative to the horizontal. Sketch the expected drag coefficient, c_d, of the diffuser as a function of α. Sketch the flow for two different diffuser angles in order to explain key features of your graph. [30%]

(b) A boat-tail is fitted at the top of the car. Sketch the expected drag coefficient, c_d, as a function of boat-tail angle and include flow sketches for different regions of your graph. [30%]

(c) Approximately what top boat-tail angle is best avoided and why? What does the flow look like for such a boat-tail? [25%]

(d) Apart from changing the drag, how else might the modifications proposed in part (a) and part (b) affect the driving behaviour at high speeds? [15%]

END OF PAPER

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1. (b) $\Omega/(\sin 2\alpha)$
1. (c) $\frac{p}{\rho} = \text{const.} - \left(\frac{\Omega r}{\sin 2\alpha}\right)^2 \left(\frac{1}{2} + \cos^2 2\alpha\right)$
1. (d) $\rho R^2\Omega^2/(2\sin\alpha)$ per unit depth
2. (a) $\Gamma/(4\pi a^2)$
2. (c) $\frac{\Gamma}{2\pi}\left(\frac{r^2}{4a^2} - \ln r_1 - \ln r_2\right)$
3. (a) $h_\infty - k/r^2$
3. (c) k/r^2
4. (b) $\frac{\pi-2}{\pi}\delta$, $\frac{4-\pi}{2\pi}\delta$, $\frac{2(\pi-2)}{4-\pi}$, $\frac{\nu\pi}{U_0\delta}$
4. (c) $\left(\frac{2\pi^2}{4-\pi}\right)^{\frac{1}{2}} \left(\frac{\nu x}{U_0}\right)^{\frac{1}{2}}$
4. (d) $0.655\left(\frac{1}{\mathrm{Re}_x}\right)$
5. (a) $\frac{0.45\nu}{U^6}\int_a^x 0.45\nu U^5 \mathrm{d}s$
5. (b) $m = 0.1125(x^4/a^4 - 1)$, $1.158a$, 0.255
6. (a) $0.87°$
6. (b) $4.35°$

Module 3A1: Fluid Mechanics I

INCOMPRESSIBLE FLOW DATA CARD

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\omega}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0, \quad \Gamma = \oint \boldsymbol{u} \cdot \boldsymbol{dl} = \int \boldsymbol{\omega} \cdot \boldsymbol{dS}$

For an irrotational flow

velocity potential ϕ $\boldsymbol{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow: $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} = \text{constant}$ throughout flow field

TWO-DIMENSIONAL FLOW

Streamfunction ψ

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

For an irrotational flow

complex potential *F(z)* $F(z) = \phi + i\psi$ is a function of $z = x + iy$

$$\frac{dF}{dz} = u - iv$$

TWO-DIMENSIONAL FLOW (continued)

Summary of simple 2 - D flow fields

	ϕ	ψ	$F(z)$	$\boldsymbol{u}$
Uniform flow (x - wise)	Ux	Uy	Uz	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	$\frac{m}{2\pi}\ln z$	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet (x - wise) at origin	$-\frac{\mu\cos\theta}{2\pi r}$	$\frac{\mu\sin\theta}{2\pi r}$	$-\frac{\mu}{2\pi z}$	$u_r = \frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = \frac{\mu\sin\theta}{2\pi r^2}$
Vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	$-\frac{i\Gamma}{2\pi}\ln z$	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\psi = 0$
Doublet at origin (with θ the angle from the doublet axis)	$-\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = \frac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = \frac{\mu\sin\theta}{4\pi r^3},\quad u_\psi = 0$

October 2015

Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U-u)u}{U^2} dy = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U^2-u^2)u}{U^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U}\right)^2\right) \frac{u}{U} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= -\frac{1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

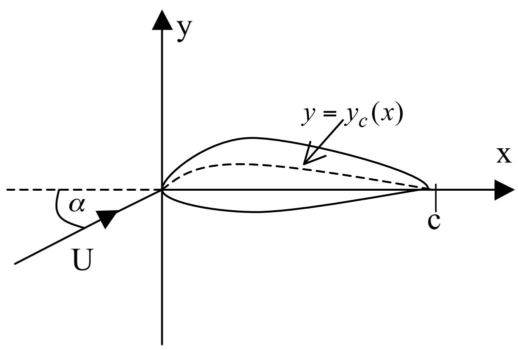
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

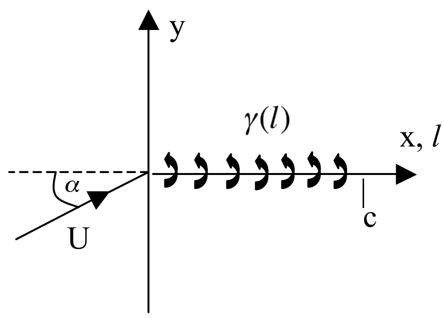
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma/U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0)/\frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $l = c(1+\cos\phi)/2, \quad x = c(1+\cos\theta)/2$

Incidence solution: $$\gamma(l) = -2U\alpha\frac{1-\cos\phi}{\sin\phi},\ c_l = 2\pi\alpha,\ c_m = c_l/4$$

Camber solution: $$\gamma(l) = -U\left[g_0\frac{1-\cos\phi}{\sin\phi} + \sum_{n=1}^{\infty} g_n \sin n\phi\right],\ \text{where}$$

$$g_0 = \frac{1}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)d\theta,\ g_n = \frac{2}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)\cos n\theta d\theta\,;$$

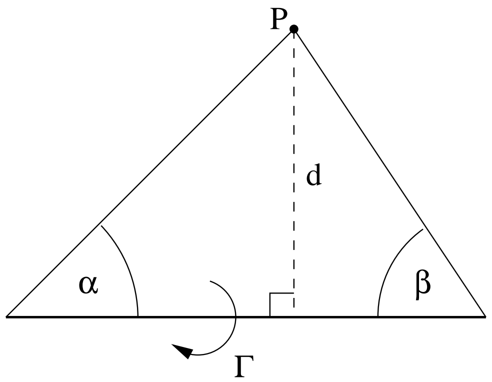
or, equivalently: $$-2\frac{dy_c}{dx} = g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\ c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{o}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta}\, d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



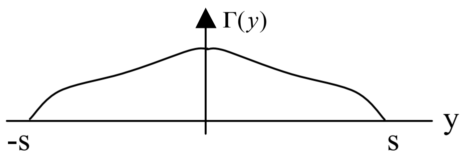
The Biot-Savart integral for a straight element of circulation Γ gives a contribution to the velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y)\, dy$

Downwash angle: $\alpha_d(y) = \dfrac{1}{4\pi U} \displaystyle\int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta}\, d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y)\alpha_d(y)\, dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{\text{n odd}} G_n \sin n\theta$, with $y = -s\cos\theta$;

equivalently, $G_n = \dfrac{2}{\pi} \displaystyle\int_{0}^{\pi} \frac{\Gamma(y)}{Us} \sin n\theta\, d\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

Elliptic lift distribution: $\Gamma(y) = \Gamma_0 \left(1 - \dfrac{y^2}{s^2}\right)^{1/2}$, $L = \dfrac{\pi}{2}\rho U \Gamma_0 s$, $\alpha_d = \dfrac{\Gamma_0}{4Us}$, $\delta = 0$