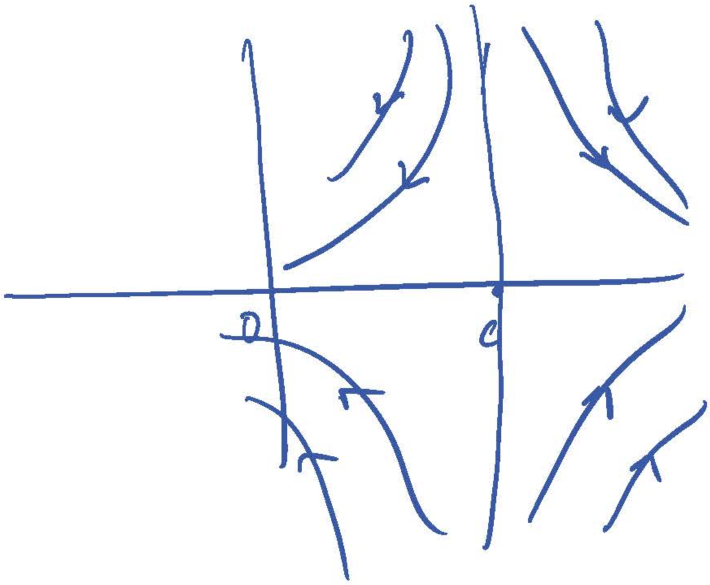
3A1_2021

1 (a) $f(z) = \frac{A}{2}(z-c)^2$



(b) $W(z) = \frac{A}{2}\left\{(z-c)^2 + \overline{\left(\frac{a^2}{\bar{z}} - c\right)^2}\right\}$

$$\therefore \frac{W(z)}{A/2} = z^2 + c^2 - 2zc + \overline{\frac{a^4}{\bar{z}^2} + c^2 - \frac{2a^2c}{\bar{z}}}$$

$$= z^2 + 2c^2 - 2zc + \frac{a^4\bar{z}^2}{|z|^4} - \frac{2a^2c\,\bar{z}}{|z|^2}$$

If $|z|^2 = a^2$

$$\frac{W(z)}{A/2} = z^2 + \bar{z}^2 + 2c^2 - 2c(z+\bar{z})$$

$$\Rightarrow \operatorname{Im}(W(z)) = 0$$

$\therefore$ $|z| = a$ is a streamline.

(c) $W(z) = \phi + i\psi$

$$\frac{W(z)}{A/2} = z^2 + \bar{z}^2\left(\frac{a^4}{|z|^4}\right) + 2c^2 - 2c\left\{z + \frac{a^2\bar{z}}{|z|^2}\right\}$$

$$\operatorname{Re}\left\{\frac{W(z)}{A/2}\right\} = (x^2-y^2)\left(1+\frac{a^4}{(x^2+y^2)^2}\right) + 2c^2$$

$$- 2cx\left\{1 + \frac{a^2}{x^2+y^2}\right\}$$

$$= (x^2-y^2)\left(1+\frac{a^4}{x^2+y^2}\right)$$

$$+ 2c(c-x)\left(1+\frac{a^2}{x^2+y^2}\right)$$

$$\therefore \phi = \frac{A}{2}\left[(1+s^2)(x^2-y^2) + 2c^2 - 2cx(1+s)\right]$$

If $r^2 = x^2+y^2$,

$$\operatorname{Im}\left\{\frac{W(z)}{A/2}\right\} = 2xy - 2xy\frac{a^4}{r^4} - 2c\left(y - \frac{a^2}{r^2}y\right)$$

$$= 2xy\underbrace{\left(1-\frac{a^4}{r^4}\right)}_{\left(1+\frac{a^2}{r^2}\right)\left(1-\frac{a^2}{r^2}\right)} - 2cy\left(1-\frac{a^2}{r^2}\right)$$

$$= 2y\left(1-\frac{a^2}{r^2}\right)\left\{x\left(1+\frac{a^2}{r^2}\right) - c\right\}$$

$$\therefore \psi = A(1-s)y\left[(1+s)x - c\right]$$

(d) $u_r = \frac{\partial \phi}{\partial r}$

$$s = \frac{a^2}{r^2} \Rightarrow \frac{\partial s}{\partial r} = -\frac{2a^2}{r^3}$$

$$x = r\cos\theta \Rrightarrow \frac{\partial x}{\partial r} = \cos\theta$$

$$\& \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\phi = \frac{A}{2}\left[(1+s^2)(x^2-y^2) + 2c^2 - 2cx(1+s)\right]$$

$$\frac{\partial \phi}{\partial r} = \frac{A}{2}\Big[(1+s^2)(2r\cos^2\theta - 2r\sin^2\theta)$$

$$+ r^2(\cos^2\theta - \sin^2\theta)\cdot 2s\left(\frac{-2a^2}{r^3}\right)$$

$$- 2c\cos\theta(1+s)$$

$$- 2cr\cos\theta\left(\frac{-2a^2}{r^3}\right)\Big]$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\therefore \frac{\partial \phi}{\partial r} = A\left[(1+s^2)\, r\cos 2\theta - \frac{2a^2}{r} s\cos 2\theta - c\cos\theta\left\{1+s - \frac{2a^2}{r^2}\right\}\right]$$

$$= A\left[(1+s^2)\, r\cos 2\theta - 2s^2 r\cos 2\theta - c\cos\theta\left\{1+s-2s\right\}\right]$$

$$= A\left[r\cos 2\theta - s^2 r\cos 2\theta - c\cos\theta\left\{1-s\right\}\right]$$

$$= A\left[r\cos 2\theta\,(1-s^2) - c\cos\theta\,(1-s)\right]$$

$\therefore u_r|_{r=a} = 0$ [we didn't need to calculate this!]

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\phi = \frac{A}{2}\left[(1+s^2)(x^2-y^2) + 2c^2 - 2cx(1+s)\right]$$

$$= \frac{A}{2}\left[r^2(1+s^2)\underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta} + 2c^2 - 2cr\cos\theta(1+s)\right]$$

$$\frac{\partial \phi}{\partial r} = \frac{A}{\cancel{2}}\left[-2r^2(1+s^2)\sin 2\theta + \cancel{2}cr\sin\theta(1+s)\right]$$

$$\therefore \frac{1}{r}\frac{\partial \phi}{\partial r} = u_\theta = A\left[-r(1+s^2)\sin 2\theta + c(1+s)\sin\theta\right]$$

$$u_\theta = A\sin\theta\left[-2(1+s^2)\cos\theta + c(1+s)\right]$$

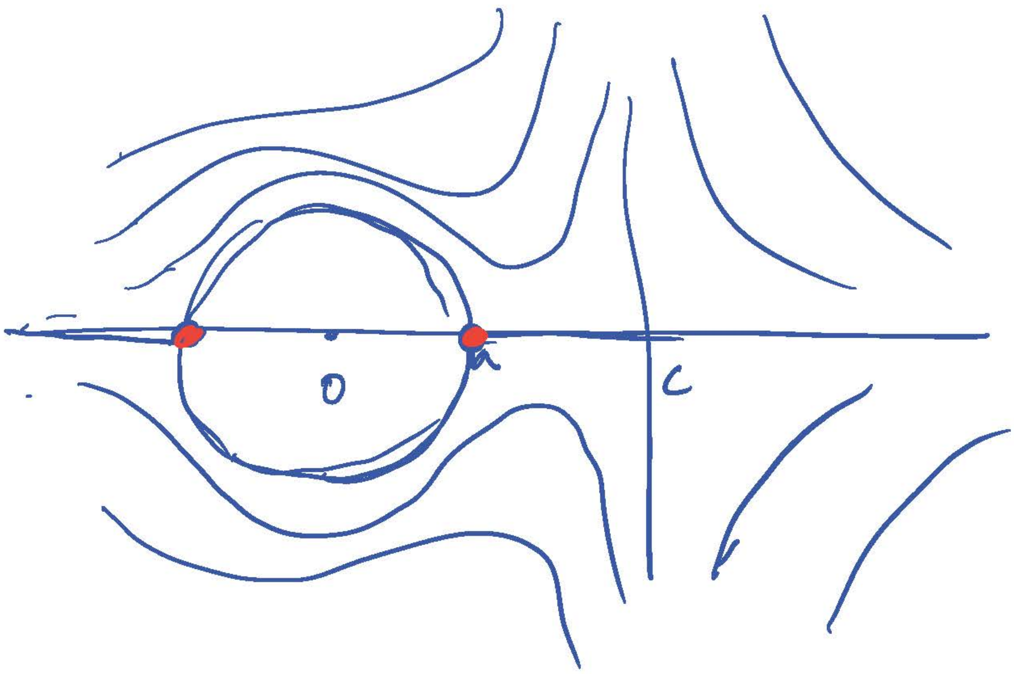
$$u_\theta\big|_{r=a} = A\sin\theta\left[-4a\cos\theta + 2c\right]$$

$$= \underbrace{2A\sin\theta}_{\mathrm{I}}\left[\underbrace{c - 2a\cos\theta}_{\mathrm{II}}\right]$$

If $c > 2a$ then II ~~is~~ is always positive.

$\therefore$ The stagnation points are

$\theta = 0, \pi$



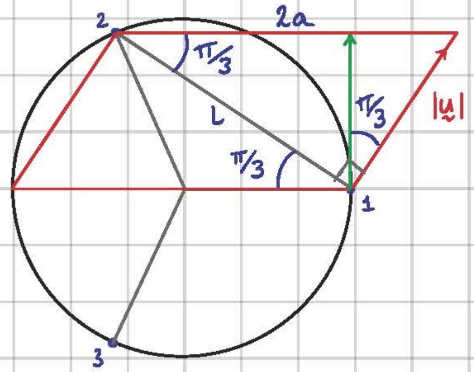
Stagnation points.

(e) Because of the symmetry of the flow about the x-axis $F_y = 0$. The resultant force will be in the $-x$ direction & it can be calculated by using the Bernoulli's equation to find and integrate the pressure on the cylinder.

2 (a) The vortices are separated by distance 2a. From the datasheet, the vertical velocity induced by each vortex on the other one is $\Gamma/4\pi a$. Therefore the vortex pair moves upwards with speed $\Gamma/4\pi a$.

(b) The vortex at $+a$ moves up while the vortex at $-a$ moves down. In polar coordinates, the radial velocity of each vortex is zero and remains zero. Therefore the vortices trace a circular arc with speed $\Gamma/4\pi a$. The radius of the arc is a, so the angular velocity is $\Gamma/4\pi a^2$ (positive).

(c) i

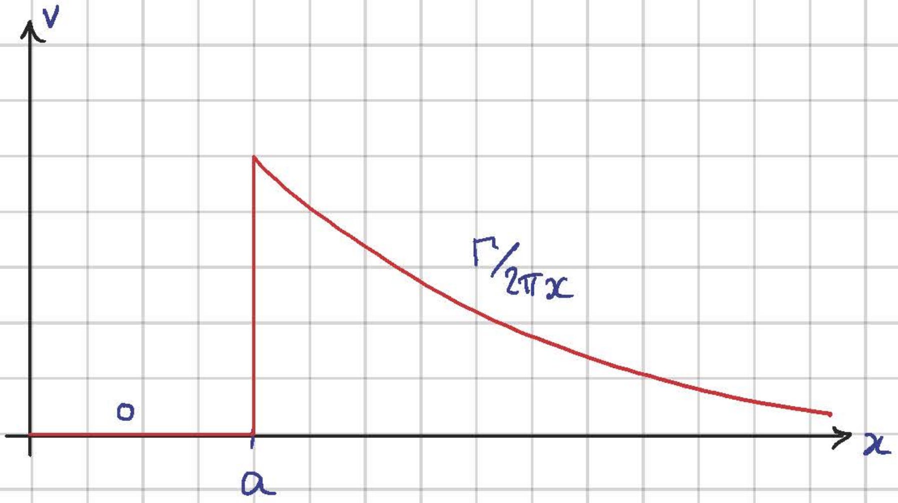


Let us consider $n = 3$ for example. This question can be answered algebraically but a geometric answer is neater. Let's consider the velocity induced at vortex 1 by vortices 2 and 3. By symmetry the horizontal velocity is zero so we need only consider the vertical velocity.

Vortex 2 is distance L from vortex 1. It induces a speed $|\underset{\sim}{u}|$ at vortex 1. The vertical component of this is $v = |\underset{\sim}{u}| \cos(\pi/3) = \dfrac{\Gamma}{2\pi} \dfrac{\cos(\pi/3)}{L}$. By inspection of the red parallelogram, $\cos(\pi/3) = L/2a$.
Therefore vortex 2 induces a vertical velocity $\Gamma/4\pi a$ at vortex 1. By symmetry, vortex 3 induces the same vertical velocity. Therefore the total induced vertical velocity is $\Gamma/2\pi a$. As for two vortices, $u_r = 0$ so the vortices trace a circular arc. The radius is a so the angular vel. is $\Gamma/2\pi a^2$.

The above argument holds for all angles (not just $\pi/3$) so, by inspection, the angular velocity induced by n line vortices, each of strength Γ/n, is $\dfrac{n-1}{n} \dfrac{\Gamma}{4\pi a^2}$.

(ii) As $n \to \infty$, the system approaches a vortex sheet with total circulation Γ. Let's consider the circulation enclosed within circular contours centred on the origin. The enclosed circulation is $\oint \underset{\sim}{u} \cdot dL = \int_0^{2\pi} u_\theta (2\pi r) d\theta$. For $r < a$, no circulation is enclosed so $u_\theta = 0$. For $r > a$, circulation Γ is enclosed so $u_\theta = \Gamma / 2\pi r$, as for a line vortex. Along the x-axis, $v = u_\theta$ so $v(x)$ is therefore:



Version SH/3

3. **Solution**

Vorticity dynamics

(a) Incompressible and irrotational flow:

$$\mathbf{u} = -\frac{1}{2}kr\hat{\mathbf{e}}_\mathbf{r} + kz\hat{\mathbf{e}}_\mathbf{z} \qquad (2)$$

$$\nabla \times \mathbf{u} = (\frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta})\hat{\mathbf{e}}_\mathbf{z} + (\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial r})\hat{\mathbf{e}}_\mathbf{r} + (\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r})\hat{\mathbf{e}}_\theta = 0 \qquad (3)$$

$$\nabla \cdot \mathbf{u} = \frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = -\frac{1}{2}\frac{\partial r(kr)}{\partial r} + \frac{\partial kz}{\partial z} = 0 \qquad (4)$$

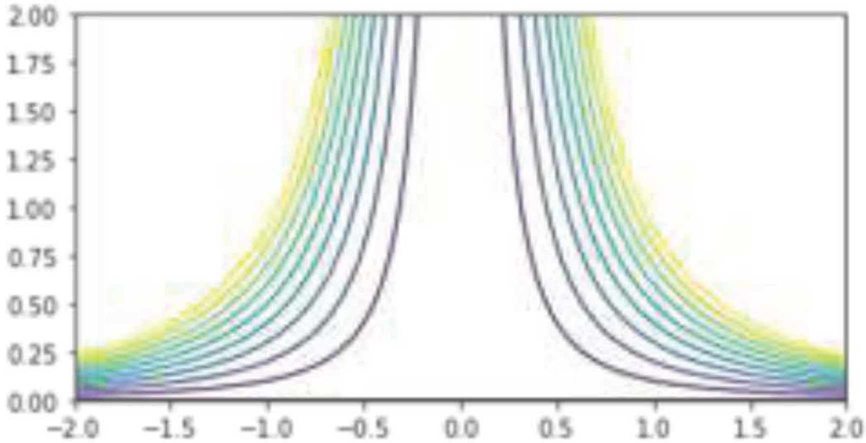
(b) Streamlines

$$\frac{dr}{u_r} = \frac{dz}{u_z} \qquad (5)$$

$$-2\frac{dr}{kr} = \frac{dz}{kz} \qquad (6)$$

$$d\ln r^{-2} = d\ln z \qquad (7)$$

$$zr^2 = \text{const.} \qquad (8)$$



(c) Vorticity conservation:

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u}$$

$$u_r\frac{\partial \omega_z}{\partial r} + u_z\frac{\partial \omega_z}{\partial z} = \omega_z\frac{\partial u_z}{\partial z}$$

The terms on the left are the convection of vorticity, and the term on the right is the stretching of the vorticity in the z direction.

$$
\begin{aligned}
\omega_r &= A r^{\alpha} z^{\alpha/2+1} \\
\frac{\partial \omega_z}{\partial r} &= A\alpha r^{\alpha-1} z^{\alpha/2+1} \\
\frac{\partial \omega_z}{\partial z} &= A(\alpha/2+1) r^{\alpha} z^{\alpha/2-1} \\
\frac{\partial u_z}{\partial z} &= k
\end{aligned}
$$

$$
-\frac{1}{2}kr\frac{\partial \omega_z}{\partial r} + kz\frac{\partial \omega_z}{\partial z} = \omega_z \frac{\partial kz}{\partial z} = \omega_z k
$$

$$
-\frac{1}{2}Akr\alpha r^{\alpha-1} z^{\alpha/2+1} + Akz(\frac{\alpha}{2}+1) r^{\alpha} z^{\alpha/2-1} = Akr^{\alpha} z^{\alpha/2+1}\left(-\frac{1}{2}+(\frac{\alpha}{2}+1)\right) = k\omega_z
$$

QED.

(d) The circulation at a given cross section z_0 is given by $\Gamma(z_0) = \oint \omega \cdot d\mathbf{S}$.

$$
\Gamma(z_0) = \int_0^{r_0} \omega_z \, 2\pi r \, dr = \int_0^{r_0} Az \, 2\pi r \, dr = \pi A r_0^{2} z_0
$$

The term $r_0^{2} z_0$ is constant along a streamtube, so the circulation is constant.

4. (a) The flow is incompressible so $\partial u/\partial x = \partial v/\partial y$. The flow is fully developed so $\partial u/\partial x = 0$, meaning that $\partial v/\partial y = 0$. But $v = -V_0$ at $y=0$ so $v = -V_0$ throughout the flow.

The (steady) momentum equation is $\quad \underline{u}\cdot\nabla\underline{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\underline{u}$

Its x-component is $\quad u\cancelto{0}{\frac{\partial u}{\partial x}} - V_0\frac{\partial u}{\partial y} = -\frac{1}{\rho}\cancelto{0}{\frac{\partial p}{\partial x}} + \nu\left(\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$

(All $\partial/\partial x \to 0$)

$$\Rightarrow \quad -V_0\frac{du}{dy} = \nu\frac{d^2u}{dy^2}$$

partial $\partial/\partial y$ collapses to ordinary d/dy because u depends only on y.

Integrate this once: $\quad -V_0\int\frac{du}{dy}\,dy = \nu\int\frac{d^2u}{dy^2}\,dy$

$$\Rightarrow \quad -V_0\Big[u\Big]_u^{U_0} = \nu\Big[\frac{du}{dy}\Big]_{(du/dy)}^{0}$$

$$\Rightarrow \quad V_0(U_0 - u) = \nu\, du/dy$$

Integrate this again $\Rightarrow \quad V_0\int_0^y dy = \nu\int_0^u\frac{du}{U_0-u} = \nu\Big[-\ln(U_0-u)\Big]_0^u$

$$\Rightarrow \quad -\frac{V_0 y}{\nu} = \ln(U_0-u) - \ln U_0 = \ln\left(\frac{U_0-u}{U_0}\right)$$

$$\Rightarrow \quad 1 - \frac{u}{U_0} = \exp\left(-\frac{V_0 y}{\nu}\right) \quad \Rightarrow \quad \frac{u}{U_0} = 1 - \exp\left(-\frac{V_0 y}{\nu}\right)$$

(b) $\quad \delta^* \equiv \int_0^\infty\left(1 - \frac{u}{U_0}\right)dy \quad$ with $\quad \frac{u}{U_0} = 1 - \exp\left(-\frac{V_0 y}{\nu}\right)$

$$= \int_0^\infty \exp\left(-\frac{V_0 y}{\nu}\right)dy = \left[-\frac{\nu}{V_0}\exp\left(-\frac{V_0 y}{\nu}\right)\right]_0^\infty = \frac{\nu}{V_0}$$

$$\tau_w = \mu\left.\frac{du}{dy}\right|_{y=0} = \mu U_0\frac{V_0}{\nu} = \rho U_0 V_0$$

(c) There is no momentum difference between the flows on the left and right boundaries. Consider streamwise length X and unit distance into the page. The mass flowrate across the top and bottom boundaries is $\rho V_0 X$. The flow arrives with speed U_0 and leaves with speed 0. Therefore the force on the fluid is the rate of change of momentum, which is $-\rho U_0 V_0 X$, per unit distance into the page. Therefore the shear stress of the fluid on the wall is $\rho U_0 V_0$, as expected.

(d) The result from part (c) does not depend on whether the boundary layer is laminar or turbulent. The result from parts (a) and (b) relies on a uniform viscosity, ν, which is not the case for a turbulent boundary layer.

5. (a) $\psi = F(x) f(\eta)$ where $\eta = \dfrac{y}{g(x)}$

ψ is the streamfunction so $u = \dfrac{\partial \psi}{\partial y} = \dfrac{F(x) f'(\eta)}{g(x)}$

In the free stream next to the bottom boundary (where $\eta \to \infty$), $\bar{U}(x) = \dfrac{Q}{x}$

Now, u must tend to $\bar{U}$ as $\eta \to \infty$ so, $\dfrac{Q}{x} = \dfrac{F(x) f'(\infty)}{g(x)}$

The streamfunction can take an arbitrary value at one point, so arbitrarily set $f'(\infty) = 1$. This gives $F(x) = Qg/x$.

(b) $u = \dfrac{Qf'}{x} = \bar{U} f'$; $\quad v = -\dfrac{\partial \psi}{\partial x} = -\dfrac{\partial}{\partial x}\left\{ \dfrac{Q g(x) f(\eta)}{x} \right\}$

$$= -Q\left\{ -\frac{g}{x^2} + \frac{1}{x} g' \right\} f + \frac{Qg}{x} f'(\eta)\left(-\frac{y}{g^2}\right) g'$$

$$= -Q\left(\frac{g'}{x} - \frac{g}{x^2}\right) f + \frac{Q}{x} y \frac{f' g'}{g}$$

$$\frac{\partial u}{\partial x} = -\frac{Q}{x^2} f' + \frac{Q}{x} f''\left(-\frac{y}{g^2} g'\right) = -\frac{Qf'}{x^2} - \frac{Q}{x} y g' \frac{f''}{g^2}$$

$$\frac{\partial u}{\partial y} = \frac{Q}{x} \frac{f''}{g} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{Q}{x} \frac{f'''}{g^2}$$

(c) $u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho} \dfrac{dp}{dx} + \nu \dfrac{\partial^2 u}{\partial y^2}$ where $\dfrac{1}{\rho}\dfrac{dp}{dx} = -\dfrac{1}{2}\dfrac{d\bar{U}^2}{dx} = -\bar{U}\dfrac{d\bar{U}}{dx} = \dfrac{Q^2}{x^3}$

$$\frac{Qf'}{x}\left(-\frac{Qf'}{x^2} - \frac{Q}{x} y g' \frac{f''}{g^2}\right) + \frac{Q}{x}\frac{f''}{g}\left(-Q\left(\frac{g'}{x} - \frac{g}{x^2}\right) f + \frac{Q}{x} y \frac{f' g'}{g}\right) = -\frac{Q^2}{x^3} + \ldots \nu \frac{Q}{x}\frac{f'''}{g^2}$$

$$\Rightarrow -\frac{Q^2 f'^2}{x^3} - \cancel{\frac{Q^2 y f' g' f''}{x^2 g^2}} - \frac{Q^2}{x}\frac{f''}{g}\left(\frac{g'}{x} - \frac{g}{x^2}\right) f + \cancel{\frac{Q^2 y f' g' f''}{x^2 g^2}} = -\frac{Q^2}{x^3} + \nu \frac{Q}{x}\frac{f'''}{g^2}$$

Divide by $-Q^2/x^3$

$$\Rightarrow f'^2 + \frac{f'' x^2}{g}\left(\frac{g'}{x} - \frac{g}{x^2}\right) f = 1 - \nu \frac{x^2}{Q}\frac{f'''}{g^2}$$

$$\Rightarrow f'^2 - \left(1 - \frac{x g'}{g}\right) f f'' = 1 - \frac{\nu x^2}{g^2 Q} f'''$$

(d) $f'^2 - \left(1 - \frac{xg'}{g}\right) f f'' = 1 - \frac{\nu x^2}{g^2 Q} f'''$

x-dependence $\qquad$ x-dependence.

For a similarity solution to exist, $\frac{xg'}{g}$ and $\frac{\nu x^2}{g^2 Q}$ must not vary in x.

① $\frac{xg'}{g} = \text{const} \Rightarrow \frac{g'}{g} = \frac{A}{x} \Rightarrow g \sim x$

② $\frac{\nu x^2}{g^2 Q} = B \Rightarrow g^2 = \frac{\nu x^2}{BQ} \Rightarrow g = \sqrt{\frac{\nu}{BQ}}\, x$

These are consistent.

Now, $\frac{g'}{g} = \frac{\sqrt{\frac{\nu}{BQ}}}{\sqrt{\frac{\nu}{BQ}}\, x} = \frac{1}{x} \Rightarrow 1 - \frac{xg'}{g} = 0$

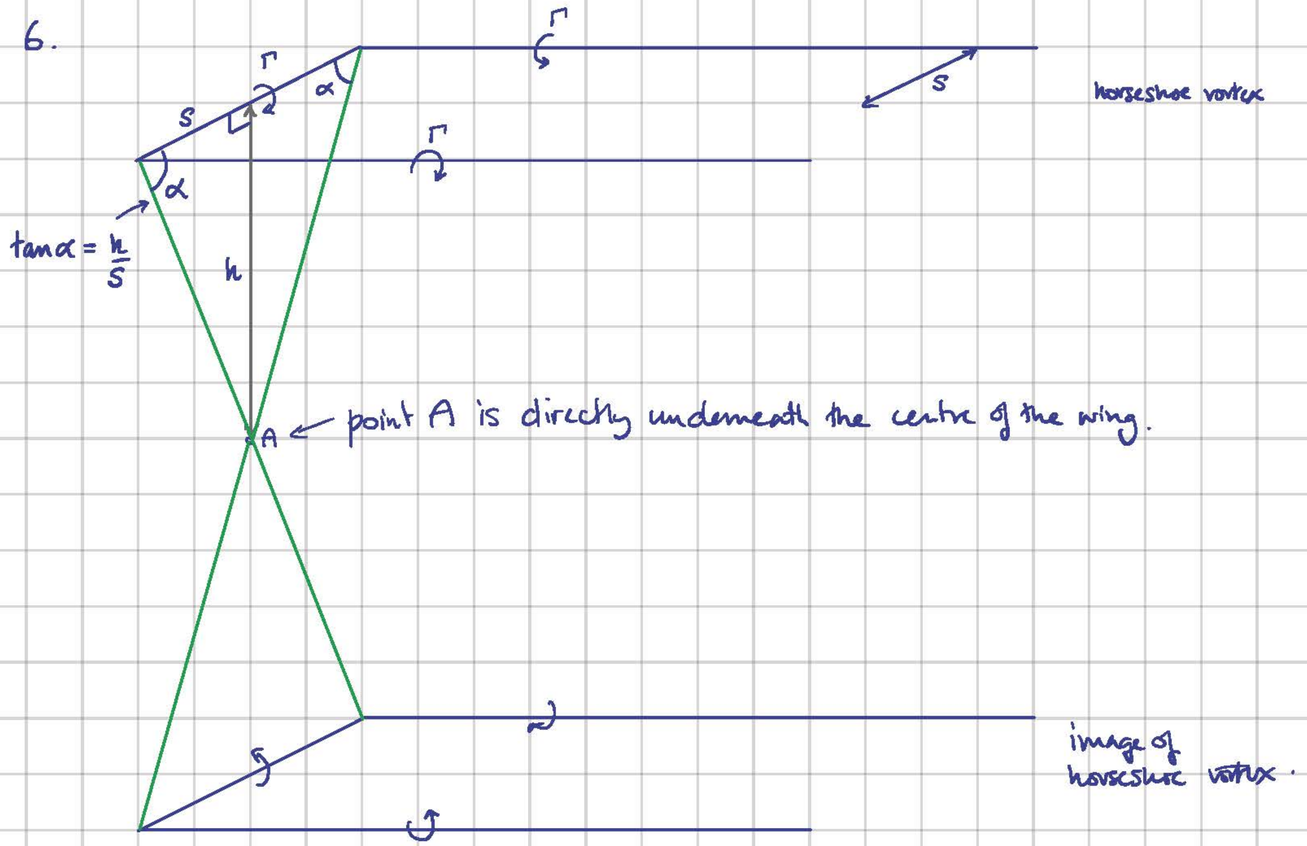
The value of B is arbitrary, so set it to 1 for simplicity.

So the similarity equation becomes $f'^2 = 1 - Bf''' \qquad \Rightarrow f'^2 = 1 - f'''$

(e) The boundary conditions are $f(0) = 0$, $f'(0) = 1$, $f'(\infty) = 1$

The boundary layers are subject to a strong adverse pressure gradient so a radial flow like this will separate around the nozzle.

6.



(a) The horizontal velocity is due only to the bound vortex because the trailing vortices' contributions cancel. The velocity at A induced by the top bound vortex is:

$$\frac{\Gamma}{4\pi h}(\cos\alpha + \cos\alpha) \quad \text{where} \quad \cos\alpha = \frac{s}{(s^2+h^2)^{1/2}}$$

If we include the bottom (image) vortex too then the horizontal velocity induced is:

$$\Delta U = \frac{\Gamma}{\pi h}\frac{s}{(s^2+h^2)^{1/2}} \quad \text{and if } h \gg s \text{ then } \Delta U \simeq \frac{\Gamma s}{\pi h^2}$$

(b) The lift force must equal W. The lift per unit length is $\rho U \Gamma$ and the effective semi-span is s. Therefore $W = 2\rho U \Gamma s \Rightarrow \Gamma = \dfrac{W}{2\rho U s}$

(c) Work in a coordinate system moving with the aircraft so that the flow is steady. The maximum velocity difference, ΔU, occurs at point A so the maximum pressure increase also occurs there. By Bernoulli's equation, $p = p_0 - \frac{1}{2}\rho U^2$
Far from A, $U = U_{aircraft}$ in this coordinate system. At A, $U = U_{aircraft} + \Delta U$.

Therefore $p_\infty - p_A = -\frac{1}{2}\rho U_{aircraft}^2 + \frac{1}{2}\rho(u_{aircraft} - \Delta u)^2$

Assuming that $\Delta u << u_{aircraft}$, $\Delta p_{max} = p_\infty - p_A \simeq \rho U_{aircraft} \Delta u$.

$$\simeq \rho U \frac{\Gamma s}{\pi h^2} \quad \text{where } \Gamma = \frac{W}{2\rho U s}$$

$$\simeq \frac{W}{2\pi h^2}$$

(d) Integrate Δp over the ground to get the force:

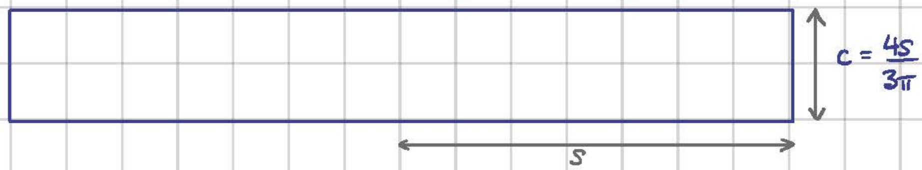
$$F = \int_{r=0}^{\infty} \Delta p \,(2\pi r\, dr) = 2\pi \Delta p_{max} \int_{r=0}^{\infty} r\left(1 + \left(\frac{r}{h}\right)^2\right)^{-3/2} dr$$

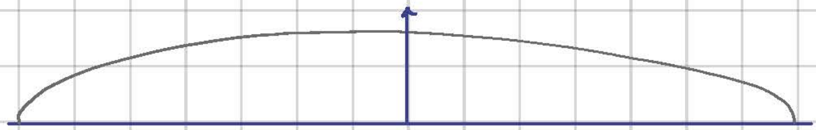
Try $\frac{d}{dr}\left\{\left(1 + \left(\frac{r}{h}\right)^2\right)^{-1/2}\right\}$. This equals $-\frac{1}{2}\left(1 + \left(\frac{r}{h}\right)^2\right)^{-3/2} \frac{2r}{h^2}$

so $F = 2\pi \Delta p_{max}\left[-h^2\left(1 + \left(\frac{r}{h}\right)^2\right)^{-1/2}\right]_0^{\infty} = 2\pi h^2 \Delta p_{max} = W$

The weight of the aircraft is carried by the ground, as is required.

7.





elliptical lift distribution
$$\Gamma(y) = \Gamma_0\left(1 - \frac{y^2}{s^2}\right)^{1/2}$$
(from datasheet)

(a) The lifting line equation with a constant chord is $\dfrac{\Gamma(y)}{\pi U c} = \alpha + \alpha_t(y) - \alpha_d$

where: α = geometric angle of attack

α_t = twist angle (function of y)

α_d = downwash angle (const.)

At the tip, $\Gamma(y=s) = 0 \Rightarrow \alpha + \alpha_t(s) = \alpha_d$

At the root, $\Gamma(y=0) = \Gamma_0 \Rightarrow \alpha + \alpha_t(0) = \alpha_d + \dfrac{\Gamma_0}{\pi U c}$

Subtracting one from the other gives $\alpha_t(0) - \alpha_t(s) = \dfrac{\Gamma_0}{\pi U c}$

But $c = 4s/3\pi \Rightarrow \Delta\alpha_t = \dfrac{3\cancel{\pi}\Gamma_0}{4s\cancel{\pi}U} = \dfrac{3\Gamma_0}{4Us}$

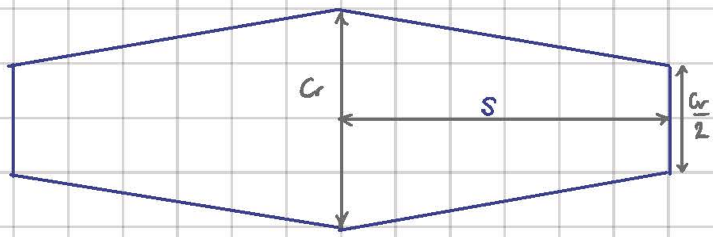
(b) At the root, $\alpha + \alpha_t(0) = \alpha_d + \dfrac{\Gamma_0}{\pi U c} = \alpha_d + \dfrac{3\Gamma_0}{4Us}$

For an elliptical lift distribution, $L = \dfrac{\pi}{2}\rho U \Gamma_0 s = W$, the aircraft weight.

and $\alpha_d = \dfrac{\Gamma_0}{4Us}$

$$\Rightarrow \alpha + \alpha_t(0) = \frac{\Gamma_0}{4Us} + \frac{3\Gamma_0}{4Us} = \frac{\Gamma_0}{Us} = \frac{2}{\pi}\frac{W}{\rho U^2 s^2} = \frac{2}{\pi}\frac{1000 \times 9.81}{1.2(50)^2(5)^2} = 0.0833 \text{ rad}$$
$$= 4.77^\circ$$

(c)



The area is the same as before, so:

$$\text{Area} = \left(Cr + \frac{Cr}{2}\right)s = \frac{4s}{3\pi} \cdot 2s$$

$$\Rightarrow \quad Cr = \frac{2}{3} \cdot \frac{4s}{3\pi} \cdot 2 = \frac{16s}{9\pi}$$

The chord is: $c(y/s) = C_r\left(1 - \frac{y}{2s}\right)$

Lifting line equation: $\dfrac{\Gamma(y)}{\pi U c(y)} = \alpha + \alpha_t - \alpha_d$

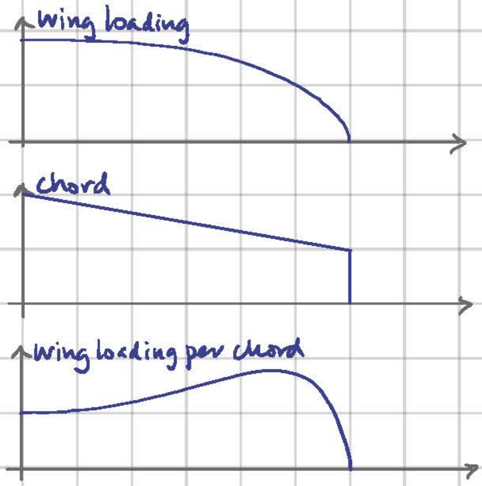
At the wing tip, $\Gamma(y=s) = 0$ so $\alpha + \alpha_t(s) = \alpha_d$ as before

At the wing root, $\alpha + \alpha_t(0) = \alpha_d + \dfrac{\Gamma_0}{\pi U Cr}$

Subtract one from the other to get $\alpha_t(0) - \alpha_t(s) = \dfrac{\Gamma_0}{\pi U Cr} = \dfrac{\Gamma_0 9\pi}{\pi U 16s} = \dfrac{9}{16}\dfrac{\Gamma_0}{Us}$

The reduction is $\left(\frac{3}{4} - \frac{9}{16}\right)\frac{\Gamma_0}{Us} = \frac{3}{16}\frac{\Gamma_0}{Us}$. This is a $\frac{3/16}{3/4} = \frac{1}{4}$ reduction compared to the straight wing.

(d) The straight wing has the highest effective angle of attack at the root and is therefore likely to stall there first. The tapered wing has a more complex wing loading per chord:

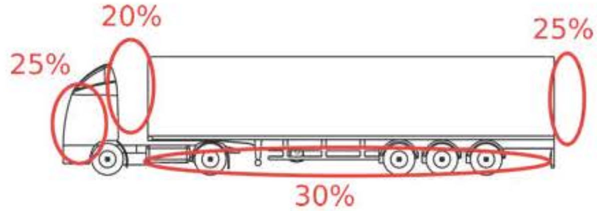


The maximum local lift/chord is further towards the tip of the wing. This can lead to tip stall, creating dangerous rolling moments if one side stalls before the other.

8. **Solution**

a)

The overriding flow features (and problems) are the separation behind the bluff trailer and the flow through the gap between the tractor and the trailer. Typically there are four major sources of aerodynamic drag: The towing vehicle (mainly the front and sides), the large wake, the tractor-trailer gap and the underbody. So figure below.



Approximate breakdown of aerodynamic drag contribution on a typical articulated HGV

b) Towing vehicles should have rounded corners and clean lines. The radiator should be kept as small as possible (the large engine requires considerable cooling so this is larger than on cars). However, practical considerations limit what can be achieved. In particular, the maximum length of a truck is limited by law (the combined length of tractor and trailer) which is why designers aim to make the towing vehicle as compact as possible to maximise load space. This is different in the US where only the length of the trailer is restricted. As a result the towing vehicle is more aerodynamic with a longer 'nose'.

The tractor-trailer gap is a significant problem. A sizeable gap is necessary to allow tight turning circles and access for maintenance and various controls (e.g. cooling on a refrigerated trailer). The best solution are spoilers on the roof of the tractor cab which direct the flow around the trailer to reduce this problem.

Tractor-trailer gap treatment

The rear of the trailer generally features no aerodynamic treatments because it is the main access for loading/unloading. Thus streamlining, which is such a domininant feature in passenger vehicle design, is completely absent. There have been experiments with add-on devices which show considerable drag reductions but these remain impractical (and illegal because of the additional length).

Modern commercial vehicles also include treatments to reduce underbody drag, similar to those found on cars. The front towing vehicle can include a small spoiler to reduce the flow towards the underside (or at least the front shape is designed to move the stagnation point down as far as possible).

Another feature widely used are side-skirts on the trailer to limit the entrainment of flow from the sides into the underbody of the vehicle – these can give significant underbody drag reductions.

Side skirts on a modern trailer[1]

However, many commercial vehicles require considerable ground clearance which limits the extent to which front spoilers and skirts can be employed.

c) For Point I –the flow separation is delayed until the trailing edge of the boat tail, and the drag is reduced accordingly.

In regime II (16 ° ≤ α ≤ 19 °), in addition to a separation bubble, strong side longitudinal vortices are formed in the near wake and rapidly increase the drag with increasing α by generating the induced drag.

In regime III (α ≥ 20 °), the main separation occurs at the leading edge of the boat tail, and thus the drag is the same as that experienced by the vehicle without the boat tail.

d)

$$\text{Fraction E saving} = 1 - \frac{{}^{1}/_{2}\rho U^{2} A\, C_{D} + R}{{}^{1}/_{2}\rho U^{2} A\, C_{D} + R}$$

Cross winds will limit applicability. There will still be an increase in skin friction and underbody drag and an increase in drag at the gaps between the vehicles.

e) Although HSTs (High Speed Trains) share many operating conditions with other ground vehicles, they exhibit some special flow features. The most obvious difference between HSTs and other ground vehicles is the large contribution of the skin friction to the total aerodynamic drag force because of the much higher length-to-width ratio of trains relative to other ground vehicles. The high speed, large length-to width ratio, and relatively low weight of an HST indicate that the flow around it causes significant instability when the train is under a strong crosswind or when trains are passing each other. When an HST enters a tunnel, the high entry speed and the large blockage ratio between the train and the tunnel cross-sectional area produce important aerodynamic phenomena that do not exist with other ground vehicles. When an HST enters a tunnel, a compression wave is generated in front of the train and propagates along the tunnel. At the tunnel exit, the compression wave is reflected back into the tunnel, forming an expansion wave. These pressure waves cause additional aerodynamic forces and moments on the train (this aerodynamic drag can exceed 90% of the total drag on a train when passing through a tunnel) and on facilities inside the tunnel. Furthermore, such waves can cause serious discomfort for passengers. The strength of the compression wave depends mainly on the blockage ratio and entry speed of an HST; by contrast, the cross-sectional shape of the train does not have a significant influence on wave generation.

Q1 Potential flow

Parts (b) and (c) of these question can be done is a couple of ways. Either by using a polar or cartesian representation of the complex quantities. In part (d), it is not necessary to calculate Ur, because by construction, on the surface of the cylinder it is zero. Most people got the location of stagnation points wrong. Note the location depends on the relationship between *c* and *a*.

Q2 Vortices

A popular question. Almost everyone got parts (a) and (b) right. Part (c) proved tricky for most candidates. There are a couple of ways to solve the problem. It can be done graphically or algebraically. Most candidates got the idea of how to solve the question right but made the trigonometry and algebra unnecessarily complicated. There was some confusion in answering c(ii). A few candidates tried to use the solution in c(i) and apply n -> infinity to obtain the graph. But in part c(i), we are asked to derive the motion of the vortices, whereas, in c(ii) we are asked to derive the vertical velocity induced along the x-axis.

Q3 Vorticity and circulation

Almost everyone got part (a) right. Several candidates tried to use a streamfunction approach to solve part (b). But streamfunctions, as described in the lectures and the databook, only applies to 2D problems. The correct (and easier) way to solve the problem is to use the definition of a streamline – curves that are tangent to the velocity vector. Part (c) was generally answered well. Some students had problems with part (d) because they were not clear on the definition and interpretation of the terms in the integral to obtain the circulation.

Q4 Boundary layer suction

The number and quality of attempts at this question were somewhat disappointing given that this was one of easiest questions. A common mistake was to assume that the vertical velocity would be zero because there is no velocity gradient in the y-direction. However, there is a vertical velocity going into the plate because of the applied suction. Those who were able to complete part (a) went on to solve part (b) correctly. There was some confusion in answering part (d).

Q5 Source Flow Boundary Layer

Popular but hard question. Most had an idea how to do it but got lost in the maths in part c. Common mistakes where to assume zero pressure gradient or apply the wrong boundary conditions. Thus, not many understood that there was a strong adverse pressure gradient which could cause separation.

Q6 Horseshoe vortex model

Not very popular question, that proved to be more difficult than expected. Common problems were that candidates treated the problem as 2-dimensional, forgot the mirror image system and did not use a moving coordinate system to make the problem steady. There were also signs that many left this question to the end and ran out of time.

Q7 Twisted Wing

Relatively popular question, well done by most candidates with quite a few near-perfect answers. Some students forgot to include the downwash when calculating the effective angle of attack. A few candidates also overlooked the fact that the datasheet contained useful information.

Q8 Lorry Aerodynamics

This was a very popular question that was well answered by many candidates. There were no particular problems and the answers discriminated well between those candidates who had a deeper understanding of the material vs those that only copied information from memory.As a general comment, there appears to be some evidence that some candidates did not make optimum use of the datasheet appended to the examination paper. A cause for this may be the difficulty of flicking back-and forth through the online paper during the examination. This may suggest that 'digital' examinations are more difficult than 'paper' versions (where the datasheet is supplied separately and can be easily consulted side-by-side with the exam paper).

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