Version AA/5

EGT2
ENGINEERING TRIPOS PART IIA

Monday 26 April 2021 9.00 to 12.10

Module 3A1

FLUID MECHANICS I

Answer not more than ***five*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachments:

- Incompressible Flow Data Card (2 pages);
- Boundary Layer Theory Data Card (1 page);
- 3A1 Data Sheet for Applications to External Flows (2 pages).

You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 30 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 (a) The complex potential for a stagnation flow is given as $F(z) = \frac{1}{2}Az^2$. Express the complex potential $f(z)$ for a stagnation flow centred at the point $z = c$, where $c > 0$ is real, and sketch the flow. [10%]

(b) The complex potential given by:

$$W(z) = f(z) + \overline{f(\frac{a^2}{\overline{z}})}$$

corresponds to the the flow of a complex potential $f(z)$, which is perturbed by the presence of a cylinder of radius a at the origin. The overline represents the complex conjugate. Using this result, obtain an expression for the complex potential for the stagnation flow centered at point $z = c$, perturbed by an added cylinder of radius $a < c$ centred at the origin, as a function of z. Show that for this specific case, $|z| = a$ is a streamline. [20%]

(c) Show that the resulting potential and streamfunctions as a function of x and y are :

$$\phi = \frac{1}{2}A\left[(1+s^2)(x^2-y^2)+2c^2-2c(1+s)x\right]$$

$$\psi = A(1-s)y\left[(1+s)x-c\right]$$

where $s = a^2/(x^2+y^2)$. [30%]

(d) Obtain the velocity components u_r and u_θ at the surface of the cylinder. Sketch the streamlines in the resulting flow, and indicate any stagnation points and singularities. [30%]

(e) Without attempting any calculations, explain how you would obtain the pressure on the surface of the cylinder, and hence the forces on the cylinder. In which direction would the resultant force act? [10%]

2 Consider a 2D inviscid incompressible flow.

(a) Two vortices are located on the $z-$plane ($z = x + iy$), one at $z = +a$ with circulation $-\Gamma$ and the other at $z = -a$ with circulation $+\Gamma$, as shown in Fig. 1 (a). Determine the velocity of the vortex pair. [20%]

(b) The sign of the circulation of the vortex at $z = +a$ is changed, as shown in Fig. 1 (b). Show that the pair rotates in a circular arc with angular velocity $\Gamma/(4\pi a^2)$. [20%]

(c) Consider n line vortices, each of circulation $+\Gamma/n$, spaced equally around a circle of radius a.

(i) Find the resulting motion of the n line vortices. [40%]

(ii) Explaining your reasoning, sketch the y-direction velocity along the $x-$axis as $n \to \infty$. [20%]

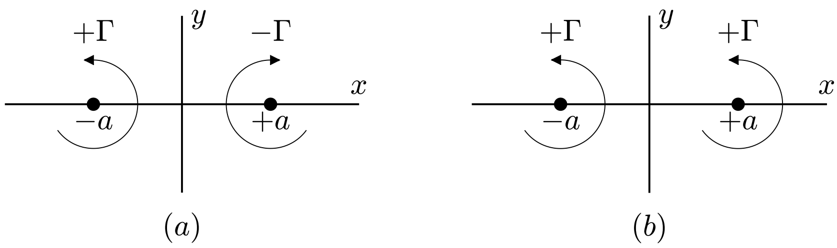

Fig. 1

3 A steady, axisymmetric flow field in 3D is given as

$$\mathbf{u} = -\frac{1}{2}kr\,\mathbf{e}_r + kz\,\mathbf{e}_z$$

in cylindrical coordinates, where k is a real constant.

(a) Show that the flow field is irrotational and incompressible. [20%]

(b) Show that the streamlines are given by $zr^2 = C$, where C is a constant. [25%]

(c) Starting from the general vorticity balance equation

$$\frac{D\omega}{Dt} = \omega \cdot \nabla\mathbf{u}$$

and the assumption that $\omega = \omega_z(r,z)\hat{\mathbf{e}}_z$, show that the equation admits solutions of the type: $\omega_z = Ar^{\alpha}z^{\alpha/2+1}$, where α is a constant. [30%]

(d) Using the solution for $\alpha = 0$, show that the circulation Γ around a streamtube that has radius r_0 when $z = z_0$ is a constant along that streamtube. [25%]

4 Consider a flat plate at zero incidence to an incompressible flow with free stream velocity U_0, density ρ, and viscosity μ, as shown in Fig. 2. The plate is porous, and fluid is removed by suction with uniform speed V_0 normal to the plate. A laminar boundary layer develops and a fully-developed regime is reached where the velocity (u, v) and pressure p are independent of x.

(a) Determine the vertical velocity v across the boundary layer. Hence integrate the x-momentum equation to find u. [30%]

(b) From the result of part (a), calculate the displacement thickness δ^* and the shear stress at the wall τ_w. [20%]

(c) Take a control volume enclosing the boundary layer (the dotted line in Fig. 2) and apply the steady flow momentum equation to find an expression for τ_w. Compare this to the expression derived in part (b). [30%]

(d) Discuss the validity of the above two approaches when the boundary layer is turbulent. Sketch the velocity profiles for laminar and turbulent boundary layers. [20%]

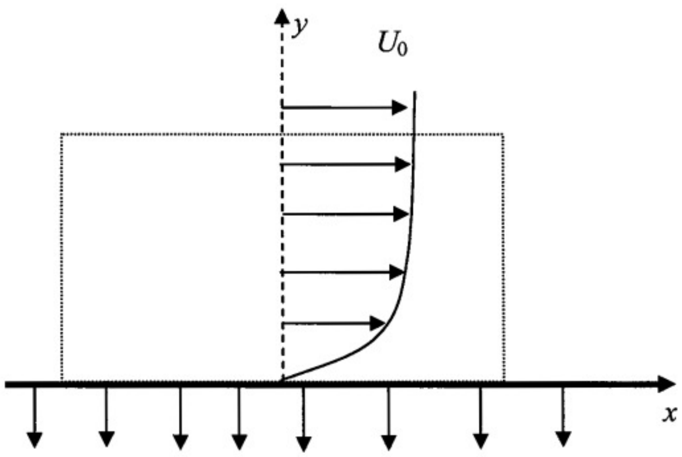

Fig. 2

5 Consider the high-Reynolds-number flow from a slit at $(x, y) = (0, 0)$ discharging between two walls at $x = 0$ and $y = 0$, as shown in Fig. 3. For this question, assume that the flow can be divided into boundary layers along the walls and an inviscid main stream with radial velocity $u_r = Q/r$, where $Q > 0$.

(a) On the horizontal boundary $y = 0$, consider a similarity solution to the boundary layer equation of the form

$$\psi = F(x) f(\eta), \quad \eta = y/g(x)$$

where ψ denotes the streamfunction. Show that $F(x)$ can be written as $Qg(x)/x$. [20%]

(b) Calculate expressions for u, v, $\partial u/\partial x$, $\partial u/\partial y$ and $\partial^2 u/\partial y^2$. [20%]

(c) Substitute the above expressions into the boundary-layer equation to deduce a differential equation relating f and g. [20%]

(d) Find the necessary conditions for the existence of a similarity solution and hence deduce the differential equation for f. [20%]

(e) State the boundary conditions for f and discuss the validity of the solution. [20%]

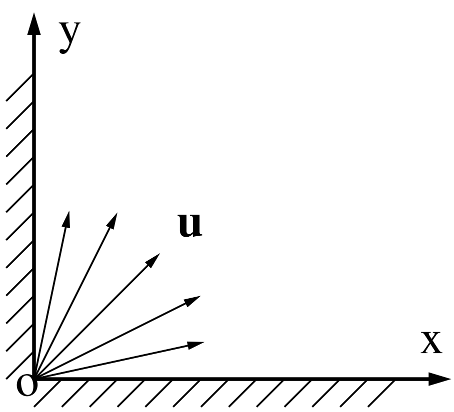

Fig. 3

6 An aircraft of weight W flying with constant horizontal velocity U at altitude h is modelled by a simple horseshoe-vortex system with effective semi-span s. You may assume that $h \gg s$ and that h is sufficiently large that any changes in the horizontal velocity at the wing can be neglected.

(a) Estimate the horizontal velocity induced on the ground directly underneath the centre of the wing. Express your answer as a function of the horseshoe vortex circulation Γ. [5%]

(b) Determine Γ as a function of W. [15%]

(c) By adopting an appropriate coordinate system, determine the maximum pressure increase on the ground, Δp_{max}, as a function of W and h. Explain your choice of coordinate system and justify your assumptions. [45%]

(d) You may assume that the pressure difference on the ground varies with radial distance r from the point directly under the centre of the wing as follows:

$$\Delta p = \Delta p_{max} \left(1 + \left(\frac{r}{h}\right)^2\right)^{-\frac{3}{2}}$$

Calculate the total force on the ground. Comment on the result. [35%]

7 An aircraft in steady horizontal flight has a rectangular wing with a chord of $c = 4s/(3\pi)$, where s is the wing semi-span. To achieve a perfect elliptical lift distribution the wing is twisted. The wing section is symmetrical.

(a) Express the total twist angle (the difference between the root and tip geometric angles of attack) as a function of the maximum wing circulation Γ_0, the flight speed U, and s. [25%]

(b) If the aircraft flies at 50 m s^{-1}, has a mass of 1000 kg, and has a semi-span of 5 m, what is the angle of attack at the wing root? The density of air is 1.2 kg m^{-3}. [25%]

(c) The wing is replaced with a tapered wing (taper ratio: 0.5) of the same area. What is the reduction in the maximum wing twist? [25%]

(d) Comment on the stall behaviour of the two wing designs. [25%]

8 (a) Identify four major sources of drag for articulated lorries. [10%]

(b) Discuss, with the aid of sketches, remedial actions for reducing drag from the sources described in (a). Discuss any limitations due to design constraints. [55%]

(c) Figure 4 plots the measured drag coefficient, C_D, against different slant angles, α, of a retrofit lorry boat tail. Briefly suggest plausible flow regimes for regions (I-III) shown in Fig. 4. [15%]

(d) Derive a simple expression for the energy savings that could be obtained for a number, N, of lorries operating in a convoy. Express the saving as a function of ρ, U, A, C_D and R where ρ is the air density, U is the lorry velocity, A is the frontal projected area, C_D is the drag coefficient, and R is the the rolling resistance. What are the aerodynamic factors that would limit energy savings? [10%]

(e) Describe the major areas where the aerodynamics of high-speed trains differ from those of lorries. [10%]

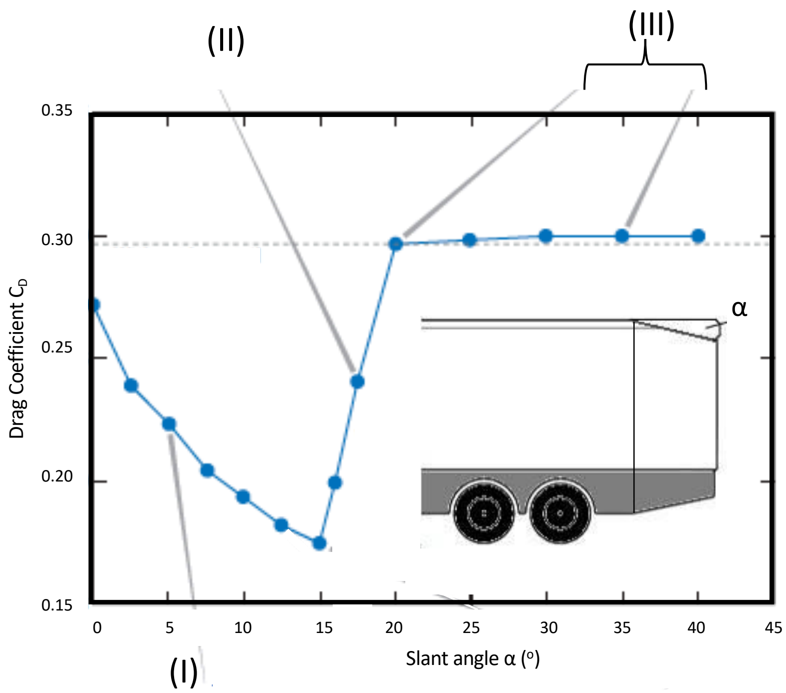

Fig. 4

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Module 3A1: Fluid Mechanics I

INCOMPRESSIBLE FLOW DATA CARD

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl } \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\omega}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0, \quad \Gamma = \oint \boldsymbol{u} \cdot \boldsymbol{dl} = \int \boldsymbol{\omega} \cdot \boldsymbol{dS}$

For an irrotational flow

velocity potential ϕ $\boldsymbol{u} = \nabla\phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow: $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} = \text{constant}$ throughout flow field

TWO-DIMENSIONAL FLOW

Streamfunction ψ

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

For an irrotational flow

complex potential *F(z)* $F(z) = \phi + i\psi$ is a function of $z = x + iy$

$$\frac{dF}{dz} = u - iv$$

TWO-DIMENSIONAL FLOW (continued)

Summary of simple 2 - D flow fields

	ϕ	ψ	$F(z)$	$\boldsymbol{u}$
Uniform flow (x - wise)	Ux	Uy	Uz	$u = U,\ v = 0$
Source at origin	$\dfrac{m}{2\pi}\ln r$	$\dfrac{m}{2\pi}\theta$	$\dfrac{m}{2\pi}\ln z$	$u_r = \dfrac{m}{2\pi r},\ u_\theta = 0$
Doublet (x - wise) at origin	$-\dfrac{\mu\cos\theta}{2\pi r}$	$\dfrac{\mu\sin\theta}{2\pi r}$	$-\dfrac{\mu}{2\pi z}$	$u_r = \dfrac{\mu\cos\theta}{2\pi r^2},\ u_\theta = \dfrac{\mu\sin\theta}{2\pi r^2}$
Vortex at origin	$\dfrac{\Gamma}{2\pi}\theta$	$-\dfrac{\Gamma}{2\pi}\ln r$	$-\dfrac{i\Gamma}{2\pi}\ln z$	$u_r = 0,\ u_\theta = \dfrac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	$\boldsymbol{u}$
Source at origin	$-\dfrac{m}{4\pi r}$	$u_r = \dfrac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\psi = 0$
Doublet at origin (with θ the angle from the doublet axis)	$-\dfrac{\mu\cos\theta}{4\pi r^2}$	$u_r = \dfrac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = \dfrac{\mu\sin\theta}{4\pi r^3},\quad u_\psi = 0$

Boundary Layer Theory Data Card

Displacement thickness;

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness;

$$\theta = \int_0^\infty \frac{(U-u)u}{U^2} dy = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy$$

Energy thickness;

$$\delta_E = \int_0^\infty \frac{(U^2-u^2)u}{U^3} dy = \int_0^\infty \left(1 - \left(\frac{u}{U}\right)^2\right) \frac{u}{U} dy$$

$$H = \frac{\delta^*}{\theta}$$

Prandtl's boundary layer equations (laminar flow);

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

von Karman momentum integral equation;

$$\frac{d\theta}{dx} + \frac{H+2}{U}\theta\frac{dU}{dx} = \frac{\tau_w}{\rho U^2} = \frac{C_f'}{2}$$

Boundary layer equations for turbulent flow;

$$\begin{aligned} \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} &= -\frac{1}{\rho}\frac{d\overline{p}}{dx} - \frac{\partial \overline{u'v'}}{\partial y} + \nu\frac{\partial^2 \overline{u}}{\partial y^2} \\ \frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} &= 0 \end{aligned}$$

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

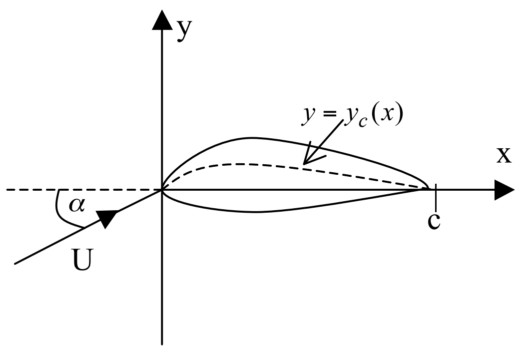
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

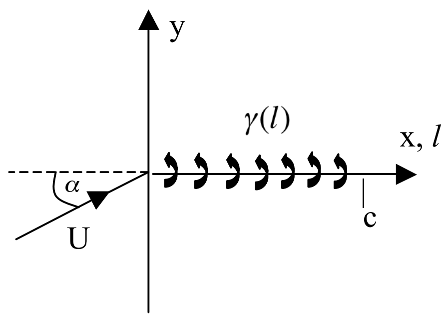
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $c_p = \pm\gamma / U$

Pitching moment coefficient: $c_m = (\text{moment about } x = 0) / \frac{1}{2}\rho U^2 c^2$

Coordinate transformation: $l = c(1+\cos\phi)/2, \quad x = c(1+\cos\theta)/2$

Incidence solution: $$\gamma(l) = -2U\alpha\frac{1-\cos\phi}{\sin\phi},\ c_l = 2\pi\alpha,\ c_m = c_l/4$$

Camber solution: $$\gamma(l) = -U\left[g_0\frac{1-\cos\phi}{\sin\phi} + \sum_{n=1}^{\infty} g_n \sin n\phi\right],\ \text{where}$$

$$g_0 = \frac{1}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)d\theta,\ g_n = \frac{2}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)\cos n\theta d\theta\,;$$

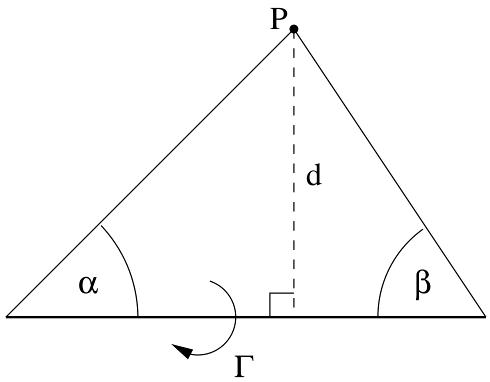
or, equivalently: $$-2\frac{dy_c}{dx} = g_0 + \sum_{n=1}^{\infty} g_n \cos n\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\ c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{o}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta} d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



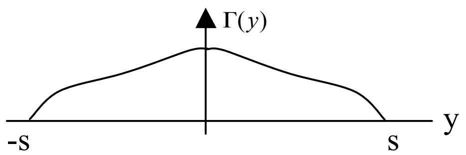
The Biot-Savart integral for a straight element of circulation Γ gives a contribution to the velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y) dy$

Downwash angle: $\alpha_d(y) = \dfrac{1}{4\pi U} \displaystyle\int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y-\eta} \, d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y) dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{\text{n odd}} G_n \sin n\theta$, with $y = -s\cos\theta$;

equivalently, $G_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} \frac{\Gamma(y)}{Us} \sin n\theta \, d\theta$

Relation between lift and induced drag:

$$C_{Di} = (1+\delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \dots$$

Elliptic lift distribution: $\Gamma(y) = \Gamma_0 \left(1 - \dfrac{y^2}{s^2}\right)^{1/2}$, $L = \dfrac{\pi}{2} \rho U \Gamma_0 s$, $\alpha_d = \dfrac{\Gamma_0}{4Us}$, $\delta = 0$