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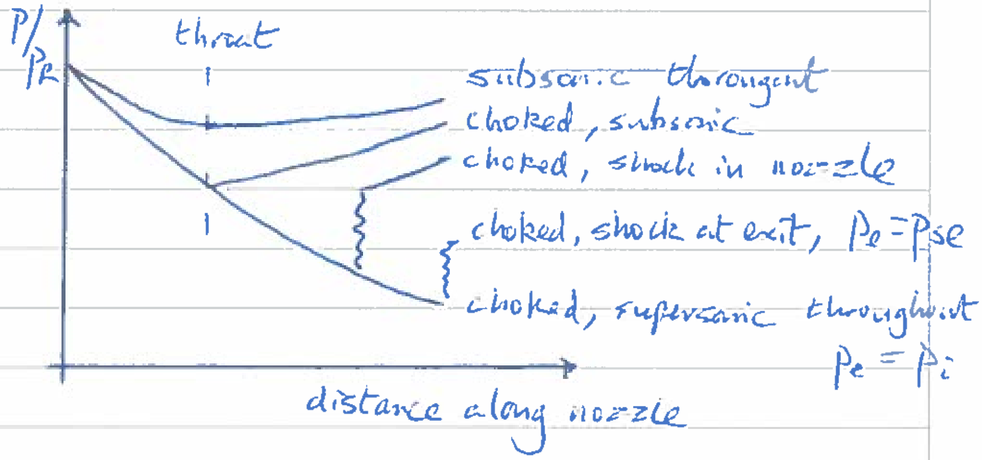
Question number 1 Sheet number 1

a) (i)

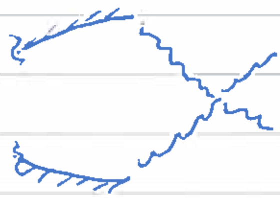
p_R : reservoir pressure

p_e : nozzle exit pressure

p_v : vessel pressure

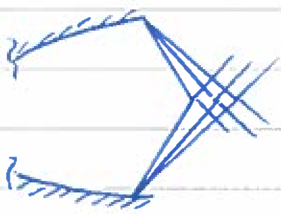


(ii) Over-expanded : $p_i < p_v < p_{se}$



shocks in exit jet

Under-expanded : $p_v < p_i$



expansions in exit jet

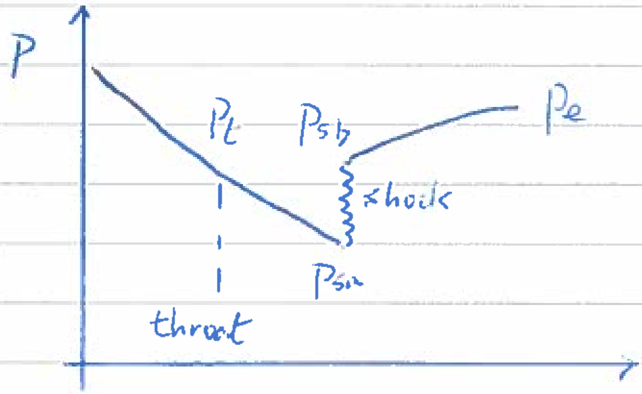
b) (i) Choked : $\dfrac{\dot m\sqrt{c_p T_0}}{A p_R} = 1.281 \Rightarrow$ exit $\dfrac{\dot m\sqrt{c_p T_0}}{A p_R} = 0.427$

Tables (subsonic) : $M = 0.195$ $\quad \dfrac{p_e}{p_R} = \underline{\underline{0.974}}$

(ii) Tables (supersonic) : $M = 2.6375$ $\quad \dfrac{p_e}{p_R} = \underline{\underline{0.0473}}$

(iii) Shock tables: $p_s/p = 7.9645$ $\quad \dfrac{p_e}{p_R} = \underline{\underline{0.377}}$

c)



Loss in p_0 $\Rightarrow$ shock present with $M = 1.89$
(and nozzle choked)

$$\frac{P_t}{P_R} = \underline{\underline{0.528}} \qquad \frac{P_{sa}}{P_R} = \underline{\underline{0.152}} \quad \text{(tables)}$$

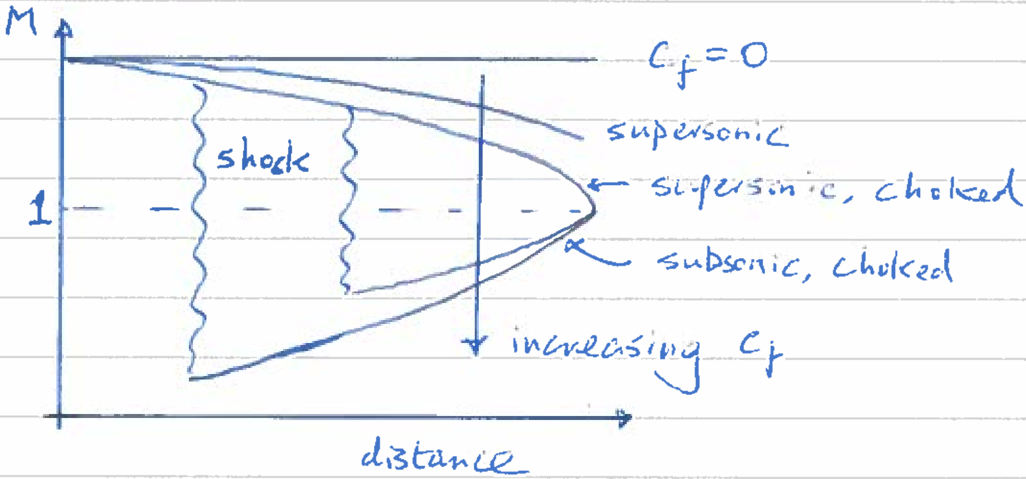
Shock: $$\frac{P_{sb}}{P_{sa}} = 4.0008 \qquad \frac{P_{sb}}{P_R} = \underline{\underline{0.608}}$$

At exit $$\frac{\dot{m}\sqrt{c_p T_0}}{A\,p_0} = \frac{\dot{m}\sqrt{c_p T_0}}{A\,p_R}\,\frac{p_R}{p_0} = 0.5531 \Rightarrow M = 0.260$$ (tables)

$$\frac{p_e}{p_0} = 0.954$$

$$\frac{p_e}{p_R} = \frac{p_e}{p_0}\,\frac{p_0}{p_R} = \underline{\underline{0.737}}$$

a)



b) (i) Exit choked : $\frac{\dot{m}\sqrt{c_p T_0}}{A p_e} = 2.4249$ (tables)

Inlet $\frac{\dot{m}\sqrt{c_p T_0}}{A p_0} = \frac{p_e}{p_0} \frac{\dot{m}\sqrt{c_p T_0}}{A p_e} = 0.8981$

	Subsonic	Supersonic	
Inlet M :	0.459	1.788	(tables)

(ii) $\frac{4 c_f L_{max}}{D}$: 1.4594 0.23796

$L_{max} = 5.9\,m$, $D = 0.2\,m$

$\Rightarrow$ $c_f = 0.0124$ or 0.0020

c)

$c_f = 0.003$

with $\dfrac{4c_f L_{max}}{D} = 0.23796$ $L_{max} = 3.966\,m$ now

At 2, for $L_{12} = 0.8\,m$ $L_{max} = 3.166\,m$ $\dfrac{4c_f L_{max}}{D} = 0.18996$

$L_{34} = 5.1\,m$

Tables: $M = 1.65$ $M_s = 0.654$ $\dfrac{4c_f (L_{max})_3}{D} = 0.314$

$(L_{max})_3 = 5.2\,m > L_{34}$. Not choked, so L_{12} too small.

Try $L_{12} = 1.0\,m$ $(L_{max})_2 = 2.966\,m$, $\left(\dfrac{4c_f L_{max}}{D}\right)_2 = 0.1780$

$L_{34} = 4.9\,m$

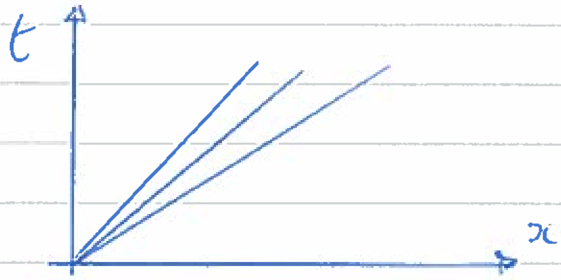
Tables: $M = 1.616$ $M_s = 0.6638$ $\dfrac{4c_f (L_{max})_3}{D} = 0.2824$

$(L_{max})_3 = 4.8\,m < L_{34}$ Predicts choked before end, so L_{12} too large.

$\therefore$ True L_{12} between $0.8\,m$ and $1.0\,m$

a) See lecture notes.

b) (i) Expansion wave system propagates rightwards from $x=0$:



(ii) $p_e = 1$ bar (e) (p) $p_p = 2.5$ bar $V_p = 0$ m/s
$T_p = 283.15$ K

$a = \sqrt{\gamma R T} \Rightarrow a_p = 337.3$ m/s

$T_e = \left(\frac{p_e}{p_p}\right)^{\frac{\gamma-1}{\gamma}} T_p$ (isentropic) $\Rightarrow T_e = 217.9$ K $a_e = 295.9$ m/s

Across R-R waves: $V_e - \frac{2}{\gamma-1} a_e = \overset{0}{\cancel{V_p}} - \frac{2}{\gamma-1} a_p$

$$V_e = -206.9 \text{ m/s}$$

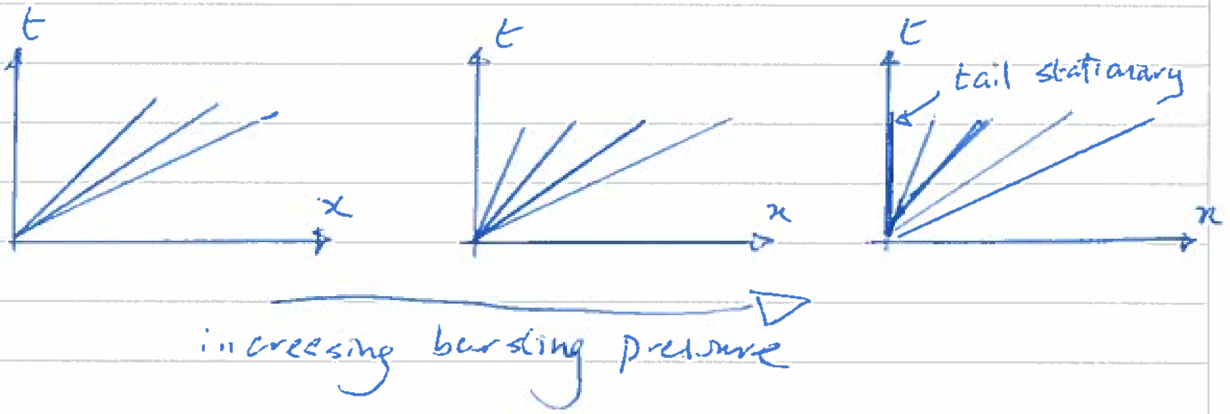
$\dot{m} = \rho_e A |V_e|$ with $\rho_e = \frac{p}{RT_e} = 1.599$ kg/m^3

$$A = 0.03142$$

$$\dot{m} = \underline{\underline{10.4}} \text{ kg/s}$$

c) (i) Higher pressure ratio ⇒ lower T_e ⇒ lower a_e

Speed of tail wave $(a_e + V_e)$ reduces.



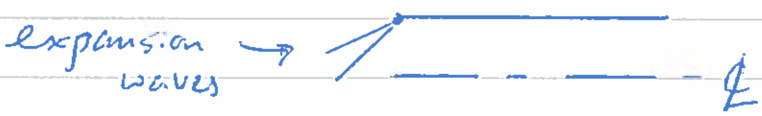
(ii) Max. possible mass flow when tail stationary (choked)

$V_e = -a_e \Rightarrow -a_e - \frac{2}{\gamma-1} a_e = -\frac{2}{\gamma-1} a_p$; $a_e = 281.1\,\text{m/s}$

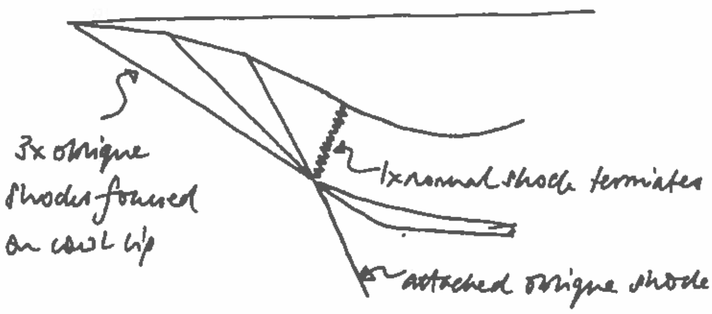
$T_e = a_e^2/\gamma R = 196.6\,\text{K}$ $\quad p_p = \left(\frac{T_p}{T_e}\right)^{\frac{\gamma}{\gamma-1}} p_e = 3.583$ bar

$\rho_e = \frac{p_e}{RT_e} = 1.772\,\text{kg/m}^3$ $\quad \dot{m} = \underline{\underline{15.65\,\text{kg/s}}}$ at $p_p = \underline{\underline{3.58\text{ bar}}}$

(iii) Further expansion in pipe not possible; exit pressure rises above ambient and expansion continues outside:



4a)(i)



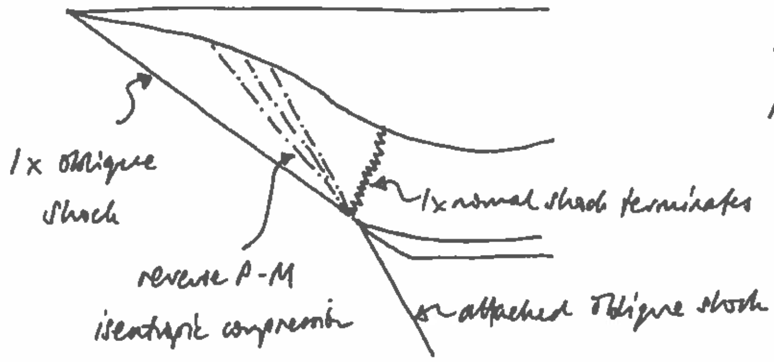
$M = 2.40 \Rightarrow \theta_{max} = 28.7°$

$3 \times 8° \Rightarrow 24°$ flow turning by oblique shocks.

$4°$ cowl lip wedge angle

↳ cowl lip @ $28°$ to free stream

∴ attached shock possible



Isentropic compression (i.e. reversed Prandtl-Meyer expansion) replaces oblique shocks 2 & 3.

ii) & iii)

M_{in}	δ	M_{out}	p_s/p	p_{os}/p_o
2.40	8°	2.08	1.63	0.988
2.08	8°	1.79	1.56	0.991
1.79	8°	1.51	1.50	0.993
1.51	normal		2.49	0.927
			9.50	90.1%

For case (b) replace shocks 2 & 3 with reversed P-M, shock 1 as before.

M	ν	p/p_o	p_{os}/p_o	p_s/p
2.08	28.6	0.113		
1.52	12.6	0.263		
1.52	normal		0.922	2.54
			91.1%	9.65

$\frac{p + \Delta p}{p} = 2.33$ [0.263/0.113]

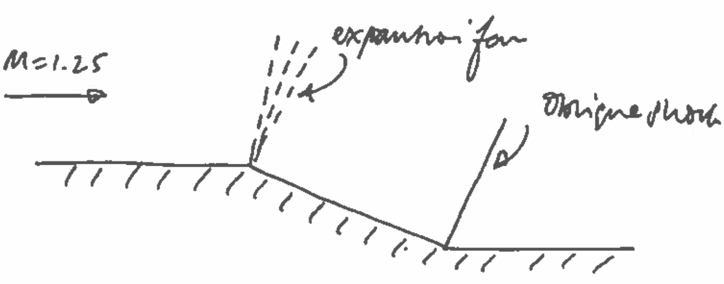
Case (b) is superior design, so use this as starting point.

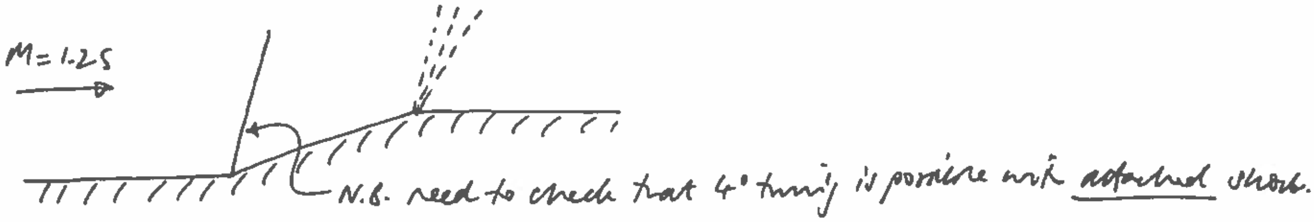
b) 1) Reduce leading ramp angle

2) Increase total turning ahead of normal shock (but note implications for cowl drag)

5a)

(i)





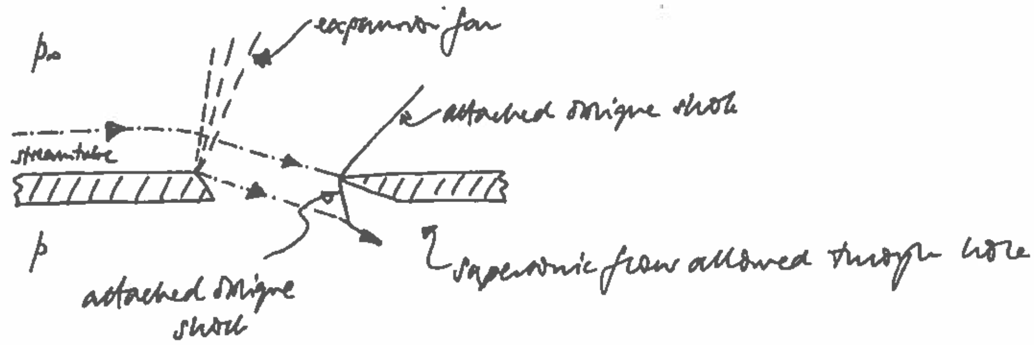
(ii) Case (a) $M = 1.25$ $\nu = 4.83$ $\Rightarrow$ 4° turning fan $\Rightarrow$ $\nu = 8.83$. $M = 1.394$, $p_0/p_{0\infty} = 0.99920$

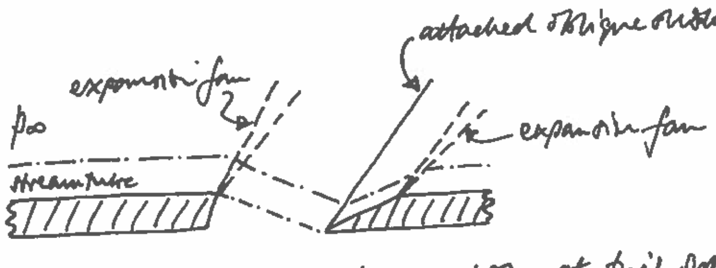
case (b) $M = 1.25$ $p_0/p_{0\infty} = 0.99882$.

(iii) Case (b) more affected as shock detaches from corner @ $M=1.15$, but remains attached in case (a) [for latter, $\nu_2 = 6.38 \Rightarrow M_2 > 1.3 \Rightarrow \theta_{max_2} > 6.662°$

b)

(i)





no flow through hole at this supersonic flow condition.

p

(ii) For a set combination of M and p/p_∞ the hole in case (d) can be "educated" to aerodynamically close, while allowing flow through in a subsonic condition. It can be used as a bleed valve with no moving parts or outside control system.

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Question number 6 Sheet number 1

a) (i) $$u_{i-1} = u_i - \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 - \frac{1}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + O(\Delta x^4)$$

$$u_{i-2} = u_i - 2\frac{\partial u}{\partial x}\Delta x + 2\frac{\partial^2 u}{\partial x^2}\Delta x^2 - \frac{4}{3}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + O(\Delta x^4)$$

so $$3u_i^n - 4u_{i-1}^n + u_{i-2}^n = 2\frac{\partial u}{\partial x}\Delta x - \frac{2}{3}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + \ldots$$

and difference eqn is

$$\frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} = -a\Delta t\left[\frac{\partial u}{\partial x} - \frac{1}{3}\frac{\partial^3 u}{\partial x^3}\Delta x^2\right]$$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{\partial^2 u}{\partial t^2}\frac{\Delta t}{2} + \frac{\partial^3 u}{\partial x^3}\frac{a\Delta x^2}{3}$$

<u>1st order</u> in time   <u>2nd order</u> in space

(ii) Points that can be made (not all required):

- For $a > 0$, this is an upwinding scheme; hence expect stability for sufficiently small $a\Delta t/\Delta x$

- $O(\Delta t)$ term can be written $-a^2\frac{\partial^2 u}{\partial x^2}\frac{\Delta t}{2}$, which represents negative diffusion ⇒ destabilising

  $O(\Delta x^2)$ term represents dispersion, which can cause oscillatory behaviour

a) (ii) cont'd • A sawtooth stability analysis, with $u_i^n = \varepsilon^n (-1)^i$, yields

$$\left|\frac{u_i^{n+1}}{\varepsilon^n}\right| = \left|1 - 4a\frac{\Delta t}{\Delta x}\right|$$

suggesting stability for $a > 0$ and $a\Delta t/\Delta x < 1/2$

b) $$u_{i+1} - u_{i-1} = 2\frac{\partial u}{\partial x}\Delta x + \frac{2}{6}\frac{\partial^3 u}{\partial x^3}\Delta x^3 + O(\Delta x^5)$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{2}{2}\frac{\partial^2 u}{\partial x^2}\Delta x^2 + O(\Delta x^4)$$

Difference eqn becomes

$$\frac{\partial u}{\partial t}\Delta t + \frac{\partial^2 u}{\partial t^2}\frac{\Delta t^2}{2} = -a\Delta t\left[\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^2}{6}\right] + \beta\frac{\partial^2 u}{\partial x^2}\Delta x^2$$

with 1st-order error $\beta\frac{\Delta x^2}{\Delta t}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\Delta t$

From underlying PDE, $\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$ to leading order

1st-order error $= \left(\beta\frac{\Delta x^2}{\Delta t} - \frac{a^2\Delta t}{2}\right)\frac{\partial^2 u}{\partial x^2}$

which is eliminated for $\underline{\underline{\beta = \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2}}$

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Question number 7a Sheet number 1

(i) Elliptic

(ii) Parabolic

(iii) Hyperbolic

See lecture notes for commentary on boundary/initial conditions and solution methods.

1/1

7b) Half Question

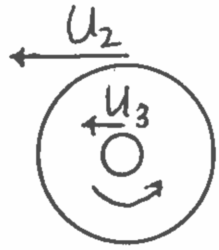
2 – rotor inlet
3 – rotor exit

$$h_{03} - h_{02} = \tfrac{1}{2}(U_3^2 - U_2^2) - \tfrac{1}{2}(V_{3,rel}^2 - V_{2,rel}^2) + \tfrac{1}{2}(V_3^2 - V_2^2)$$

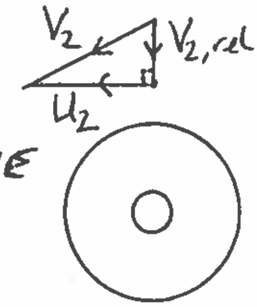
large power output $\Rightarrow$ $h_{03} - h_{02}$ large negative.

(i) $\frac{1}{2}(U_3^2 - U_2^2)$ LARGE NEGATIVE $\Rightarrow U_3 < U_2$

$\Rightarrow$ RADIAL INFLOW TURBINE



(ii) $-\frac{1}{2}(V_{3,rel}^2 - V_{2,rel}^2)$ LARGE NEGATIVE



$\Rightarrow V_{2,rel}^2$ SMALL AS POSSIBLE

$\Rightarrow$ RADIAL FLOW IN RELATIVE FRAME

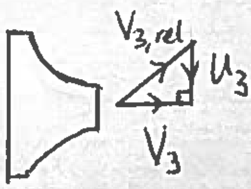
$\Rightarrow \alpha_{2,rel} = 0$, $V_{\theta 2} = U_2$

(iii) $\frac{1}{2}(V_3^2 - V_2^2)$ SMALL AS POSSIBLE, ie negative

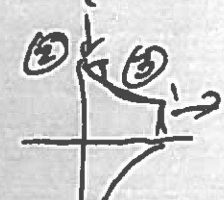
$\Rightarrow V_3^2$ MINIMUM

$\Rightarrow \alpha_3 = 0$

NO EXIT SWIRL $V_{\theta 3}$



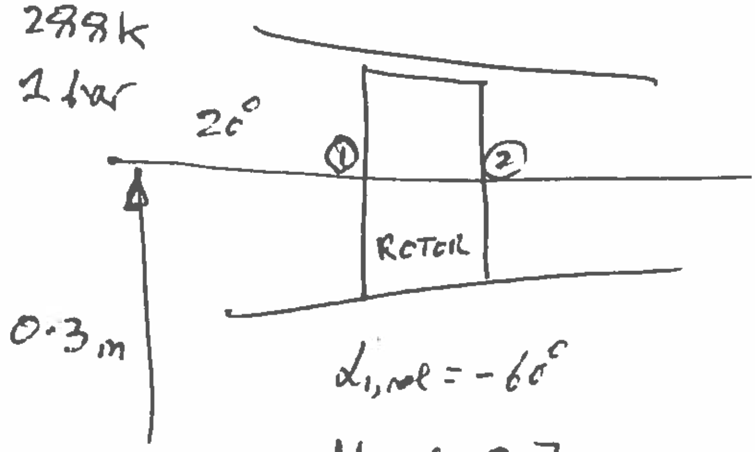
(iv) $\psi = \dfrac{U_2 V_{\theta 2} - U_3 V_{\theta 3}}{U_2^2} = \dfrac{U_2 . U_2 - 0}{U_2^2} = 1$



N.B. Also derivable from expression given for Δh_0, using velocity triangles as sketched.

8)

①/④



$v_x = \text{const.}$

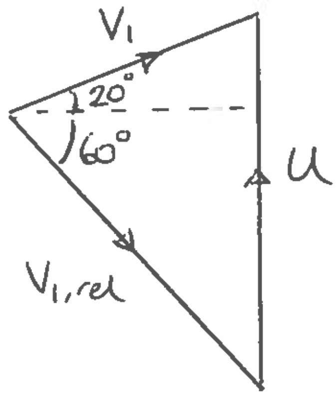
$\alpha_{1,rel} = -60°$

$M_{1,rel} = 0.7$

$Y_p = 0.04$

$\alpha_{2,rel} = -30°$

a)

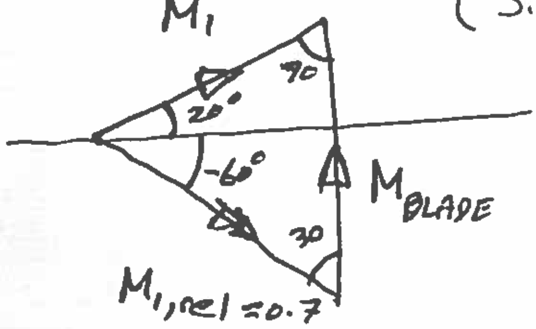


b) $\tan \alpha_{rel} = \tan \alpha - \frac{1}{\phi}$

$$\phi = \left(\tan \alpha - \tan \alpha_{rel}\right)^{-1}$$

$$\phi = \left(\tan 20° - \tan(-60°)\right)^{-1} = \underline{\underline{0.4771}}$$

c)



(Similar to velocity Δ.)

$$\frac{M_{1,rel}}{\sin 70°} = \frac{M_1}{\sin 30°} \qquad M_1 = \underset{0.7}{M_{1,rel}} \frac{\sin 30°}{\sin 70°}$$

$$\underline{\underline{M_1 = 0.3725}}$$

(d) $\gamma = 1.4 \quad R = 287\ J/kgK$ ② / ④

$C_p = 1005\ J/kgK$

$$\frac{T_{01}}{T_1} = \left(1 + \frac{\gamma-1}{2} M_1^2\right)$$

$$T_1 = \frac{T_{01}}{1 + \frac{\gamma-1}{2} 0.3725^2} = \frac{288}{1.0278} = \underline{\underline{280.2\ K}}$$

$$P_1 = \left(\frac{T_1}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} P_{01} = 1\ bar \left(\frac{280.2}{288.0}\right)^{3.5} = \underline{\underline{0.9084\ bar}}$$

$$V_1 = M_1 \sqrt{\gamma R T_1} = 0.3725 \sqrt{1.4 \times 287 \times 280.2}$$

$$\underline{V_1 = 124.9\ m/s}$$

$$V_{x1} = V_1 \cos \alpha_1 = 124.9 \times \cos 20^\circ = \underline{\underline{117.4\ m/s}}$$

$$V_{\theta 1} = V_1 \sin \alpha_1 = 124.9 \times \sin 20 = \underline{\underline{42.7\ m/s}}$$

(e) $\phi = \frac{V_x}{U} \Rightarrow U = \frac{V_x}{\phi} = \frac{117.4}{0.4771} = \underline{\underline{246.1\ m/s}}$

$U = r\Omega \Rightarrow \Omega = \frac{U}{r} = \frac{246.1}{0.3} = \underline{\underline{820.3\ rad/s}}$

(7833.6 rpm)

(f)

$$\frac{T_{01,rel}}{T_1} = \left(1+\frac{\gamma-1}{2} M_{1,rel}^2\right)$$

$$T_{01,rel} = 280.2\left(1+\frac{\gamma-1}{2}0.7^2\right) = 280.2 \times 1.098$$

$$= \underline{\underline{307.7\,k}}$$

$$P_{01,rel} = P_1\left(\frac{T_{01,rel}}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 0.9084 \times (1.098)^{3.5} = \underline{\underline{1.2600\text{ bar}}}$$

Flow as fixed/constant radius

$$\Rightarrow T_{02,rel} = T_{01,rel} = \underline{307.7\,k}$$

LOSS COEFFICIENT

$$0.04 = Y_p = \frac{P_{01,rel} - P_{02,rel}}{P_{01,rel} - P_1} \quad \text{(COMPRESSOR)}$$

$$P_{02,rel} = P_{01,rel} - 0.04(P_{01,rel} - P_1)$$

$$= 1.2600 - 0.04(1.2600 - 0.9084)$$

$$\underline{P_{02,rel} = 1.2459\text{ bar}}$$

$$V_{x2} = V_{x1} = 117.4\text{ m/s}$$

$$V_{\theta 2,rel} = V_{x2}\tan\alpha_{2,rel} = 117.4\tan(-30^\circ)$$

$$\underline{V_{\theta 2,rel} = -67.8\text{ m/s}}$$

(e) cont.

$$T_2 = T_{02,rel} - \frac{\left(v_{x2}^2 + v_{\theta 2,rel}^2\right)}{2c_p}$$

$$T_2 = 307.7 - \frac{\left(117.4^2 + (-67.8)^2\right)}{2 \times 1005} = \underline{298.6\ \text{K}}$$

$$p_2 = p_{02,rel}\left(\frac{T_2}{T_{02,rel}}\right)^{\frac{\gamma}{\gamma-1}} = 1.2459\left(\frac{298.6}{307.7}\right)^{3.5} = \underline{1.1216\ \text{bar}}$$

$$\Rightarrow \quad \frac{p_2}{p_1} = \frac{1.1216}{0.9084} = \underline{\underline{1.235}}$$

(f) g

$$v_{\theta 2} = U + v_{\theta 2,rel} = 246.1 - 67.8 = \underline{178.3\ \text{m/s}}$$

$$\text{stage loading} = \frac{\Delta h_0}{U^2} = \frac{U\left(v_{\theta 2} - v_{\theta 1}\right)}{U^2}$$

$$\psi = \frac{178.3 - 42.7}{246.1} = \underline{\underline{0.55}}$$

This is rather high, would expect $\psi \approx 0.4$