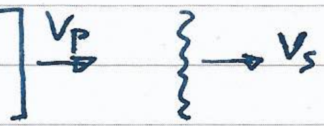
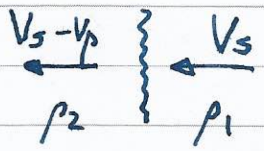
a)



Stationary fluid

Shock frame:



Continuity: $\rho_2 (V_s - V_p) = \rho_1 V_s$; $\underline{\underline{\dfrac{\rho_2}{\rho_1} = \dfrac{V_s}{V_s - V_p}}}$

b) Assume $V_s = 497$ m/s, giving Mach no 1.461

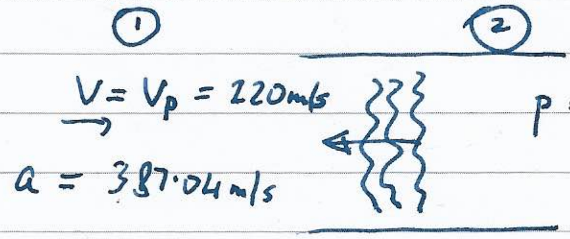
LHS of continuity eqn: $\dfrac{\rho_2}{\rho_1} = \dfrac{p_2}{p_1}\dfrac{T_1}{T_2} = 2.3236 \times 1.2945 = \underline{\underline{3.008}}$

while ~~RH~~ RHS $\dfrac{V_s}{V_s - V_p} = \underline{\underline{1.79}}$ Hmm.

Assuming $V_s = 497$ m/s anyway, $p_1 = \underline{\underline{2.32 \times 10^5 \text{ Pa}}}$

$T_1 = \underline{\underline{373 \text{ K}}}$

c)



$p = 10^5$ Pa

What else??

Ans: isentropic across expansion waves.

$$a_2 = a_1 \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{2\gamma}} = 343.12 \text{ m/s}$$

and $V_1 + \dfrac{2a_1}{\gamma - 1} = V_2 + \dfrac{2a_2}{\gamma - 1} \Rightarrow V_2 = \underline{\underline{439.6 \text{ m/s}}}$

$T_2 = \underline{\underline{293.0 \text{ K}}}$

d) At time T, piston is @ $x = x_T$, $x_T = V_p T$

Time for shock to reach end: $t_s = L/V_s$

Head of expansion travels ← at $a_1 - V_p$

Time to reach $x = x_T$ is $\dfrac{L - x_T}{a_1 - V_p}$

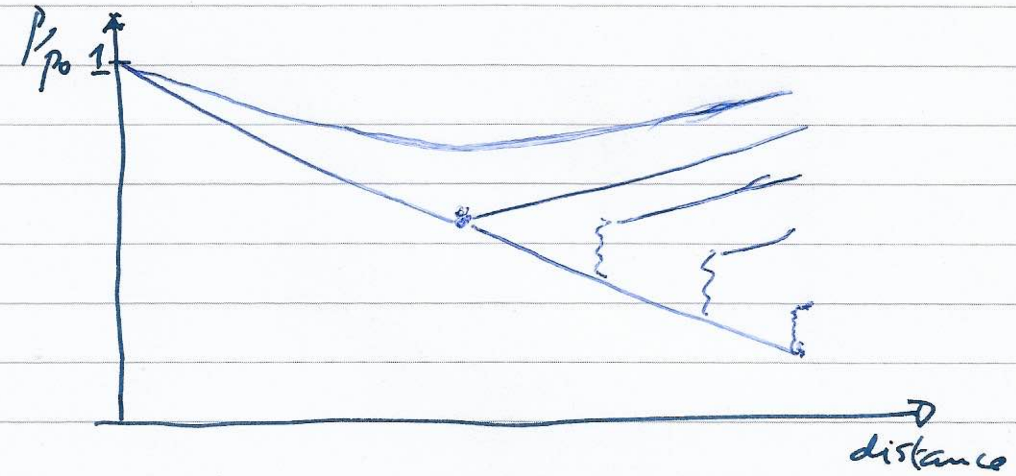
so $$\frac{L}{V_s} + \frac{L - x_T}{a_1 - V_p} = T = \frac{x_T}{V_p}$$

$$\frac{L}{x_T} \cancel{\frac{x_T}{L}} = \left(\frac{1}{V_p} + \frac{1}{a_1 - V_p}\right) \Big/ \left(\frac{1}{V_s} + \frac{1}{a_1 - V_p}\right)$$

$$= \underline{1.32}$$

$$\frac{x_T}{L} = \underline{0.76}$$

a)



$\frac{p_{0e}}{p_0} = 1$ — when flow subsonic throughout
— when flow supersonic and shock-free downstream of throat.

Otherwise, $p_{0e}/p_0 < 1$ due to presence of shock.

If $\frac{p_{0e}}{p_0} = 0.939$, then Mach no. upstream of shock is 1.470

and $\frac{\dot{m}\sqrt{c_pT_0}}{A_s p_0} =$ ~~1.0894~~ 1.1077

Meanwhile $\frac{\dot{m}\sqrt{c_pT_0}}{A_t p_0} = 1.2810$

$\Rightarrow \frac{A_s}{A_t} = \underline{\underline{1.156}}$

b) Downstream of shock $\frac{\dot{m}\sqrt{c_pT_0}}{A_s p_{0s}} = \frac{1.1077}{0.939} = 1.180$

At exit, with $A_e = 1.2A_t = 1.2\frac{A_s}{1.156} = 1.038A_s$

$\frac{\dot{m}\sqrt{c_pT_0}}{A_e p_{0s}} = 1.1364$ $M_e = \underline{\underline{0.659}}$

$\underline{\underline{\frac{p_e}{p_0}}}$ $M = 0.659 \Rightarrow \frac{p_e}{p_{0s}} = (1.3387)^{-1} = 0.7470$

$$\frac{p_e}{p_0} = \frac{p_e}{p_{0s}} \frac{p_{0s}}{p_0} = \underline{0.701}$$

c) Let length of diverging section be L

Linear relationship: $L = 5 \frac{A_e - A_t}{A_t} \times L$

More generally $x = 5 \frac{A - A_t}{A_t} \times L$

In (b) $x = 0.78L \Rightarrow$ now shock is @ $0.98L$
and $A = 1.196 A_t$

Upstream of shock: $\frac{\dot{m}\sqrt{c_p T_0}}{A p_0} = 1.071$; $M = 1.5286$

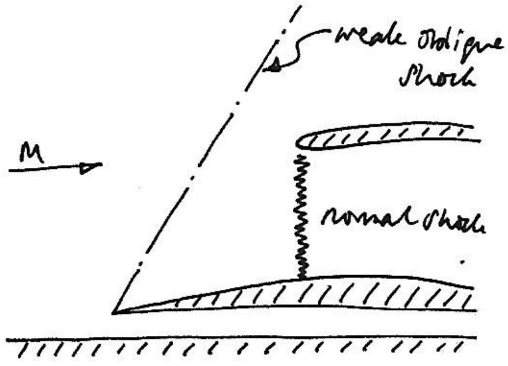
Downstream: $\frac{p_{0s}}{p_0} = 0.92046$ $\frac{\dot{m}\sqrt{c_p T_0}}{A p_{0s}} = 1.16355$

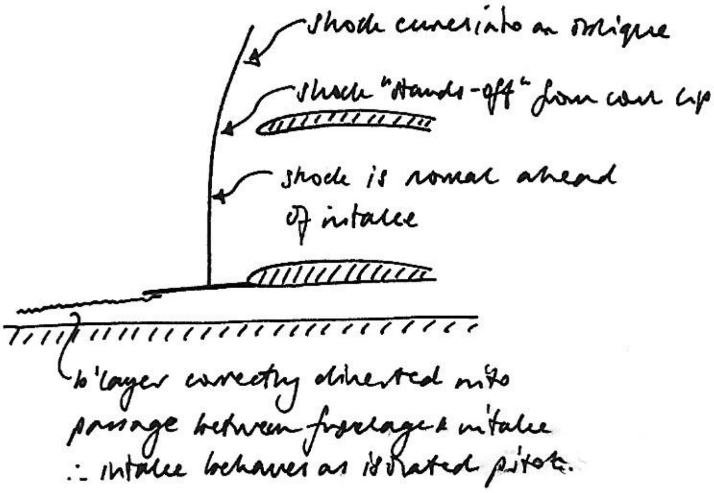
At exit $\frac{\dot{m}\sqrt{c_p T_0}}{A_e p_{0s}} = \frac{1.196}{1.2} \frac{\dot{m}\sqrt{c_p T_0}}{A p_{0s}} = 1.1597$

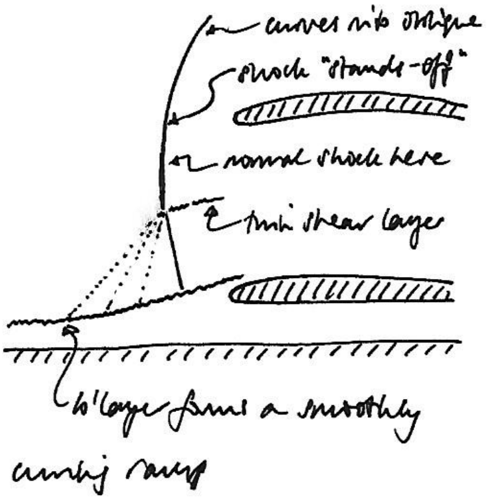
$M_e = 0.6864$, $\frac{p_e}{p_{0s}} = 0.7296$ $\frac{p_e}{p_0} = 0.6716$

$\underline{\underline{\text{4.2\% decrease}}}$

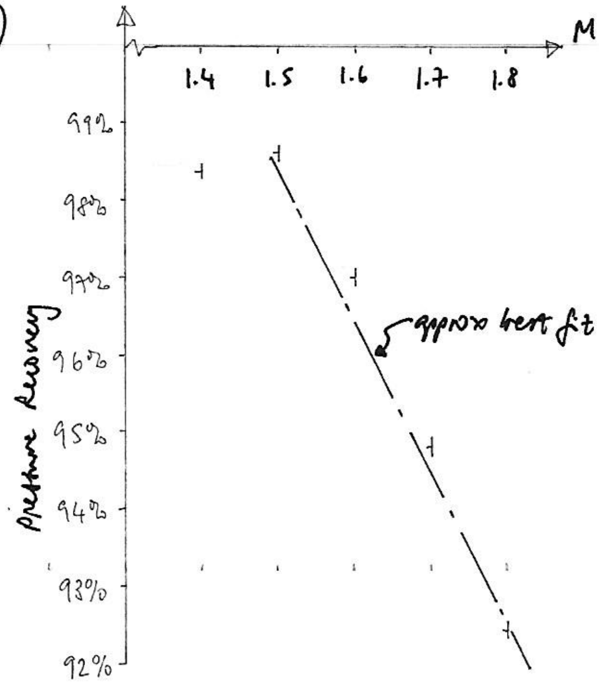
3a)







b)



c) Pressure recovery of intake (b) is 0.984 @ $M = 1.40$.

Normal shock loss @ $M = 1.40 = 0.958$

Assume lower half of intake (c) is essentially isentropic gives overall pressure recovery = 0.979.

Requires engine to have sufficient fan/compressor surge margin to tolerate the associated distortion.

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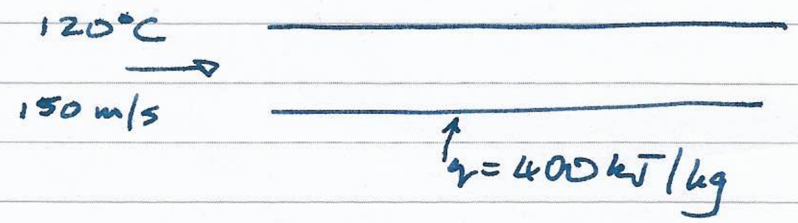
Question number 4 Sheet number 1

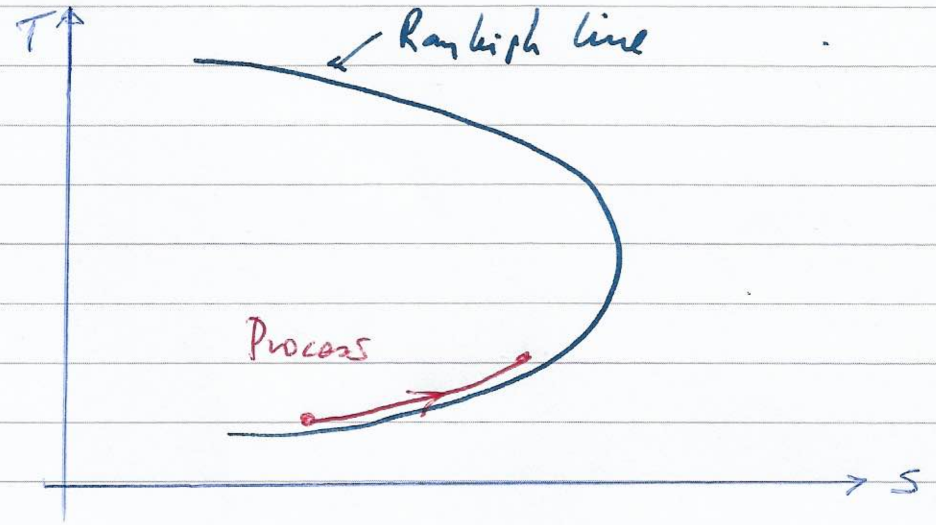


a) Inlet: $a = 397.45$ m/s $\quad M = \underline{0.377} \quad \dfrac{F}{\dot{m}\sqrt{c_p T_0}} = 1.4169$

Outlet: $\Delta T_0 = \dfrac{q}{c_p} = 398.01\,\text{K}$ and $F = F_{inlet}$

$$\Rightarrow \frac{F}{\dot{m}\sqrt{c_p T_0}} = 1.4169\sqrt{\frac{404.34}{802.35}} = 1.0058; \quad M = \underline{0.811}$$

b)



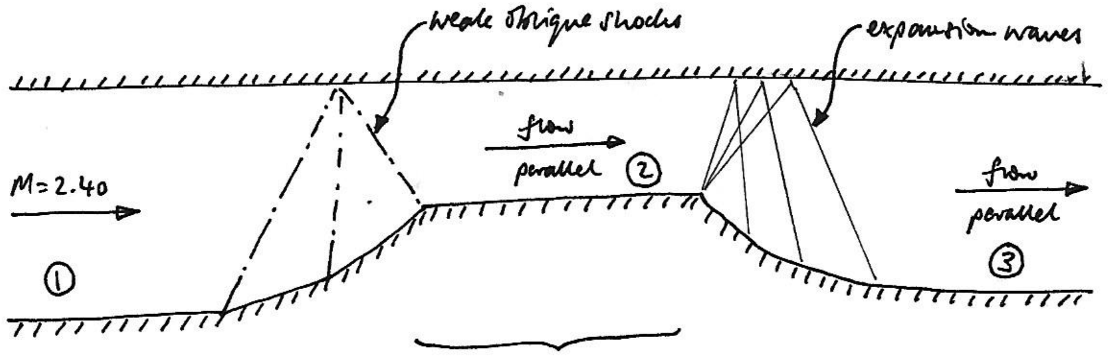
c) At some point, exit becomes choked. Higher q than this would imply M = 1 before exit, precluding further heat addition and contradicting assumption of higher q.

Max rate when exit choked, with $\dfrac{F}{\dot{m}\sqrt{c_p T_0}} = 0.9897$

$$\frac{(T_0)_{out}}{(T_0)_{in}} = \left(\frac{1.4169}{0.9897}\right)^2 = 2.050 \qquad (T_0)_{out} = 828.74\,\text{K}$$

$$q = c_p \Delta T_0 = \underline{426.5\ \text{kJ/kg}}$$

5 a)



b) Consider the 3 weak obliques; from tables:

M_1	θ	M_2	p_{02}/p_{01}	p_2/p_1
2.40	10°	2.00	0.978	1.83
2.00	6°	1.79	0.996	1.39
1.79	16°	1.18	0.947	2.26
			0.922	5.75

$\Rightarrow M_2 = 1.18$

c) $p_2/p_1 = 5.75$

d) $M_2 = 1.18 \Rightarrow \nu_2 = 3.07$ from tables

Initial fan turns flow through 16°, reflected fan another 16°

$\Rightarrow \nu_3 = 35.07 \Rightarrow M_3 = 2.33$

e) From tables, $p_1/p_{01} = 0.0684$ and $p_3/p_{03} = 0.0763$

Fans are isentropic $\therefore p_{03} = p_{02} \Rightarrow p_{03}/p_{01} = p_{02}/p_{01}$

$$p_3/p_1 = \frac{p_3}{p_{03}} \times \frac{p_{03}}{p_{01}} \times \frac{p_{01}}{p_1} = \frac{0.0763}{0.922 \times 0.0684} = 1.21$$

Constriction has slowed flow (M reduced) and raised static pressure.

6 a)

$$\frac{\partial T}{\partial x} = \frac{\alpha}{u}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial x} = \frac{(T_i^{j+1} - T_i^j)}{\Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{(T_i^{j+1} - 2T_i^j + 2T_i^{j-1})}{\Delta x^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{(T_{i+1}^j - 2T_i^j + 2T_{i-1}^j)}{\Delta \mathrm{y}^2}$$

Therefore the finite difference approximation is

$$\frac{(T_i^{j+1} - T_i^j)}{\Delta x} = \frac{\alpha}{u}\left(\frac{(T_{i+1}^j - 2T_i^j + 2T_{i-1}^j)}{\Delta \mathrm{y}^2} + \frac{(T_i^{j+1} - 2T_i^j + T_i^{j-1})}{\Delta x^2}\right)$$

$$(T_i^{j+1} - T_i^j) = \frac{\alpha \Delta x}{u}\left(\frac{(T_{i+1}^j - 2T_i^j + 2T_{i-1}^j)}{\Delta \mathrm{y}^2} + \frac{(T_i^{j+1} - 2T_i^j + T_i^{j-1})}{\Delta x^2}\right)$$

$$(T_i^{j+1} - T_i^j) = \sigma T_{i+1}^j - 2\sigma T_i^j + \sigma T_{i-1}^j + \gamma T_i^{j+1} - 2\gamma T_i^j + \gamma T_i^{j-1}$$

With $\sigma = \frac{\alpha \Delta x}{u \Delta y^2}$ and $\sigma = \frac{\alpha}{u \Delta \mathrm{x}}$

Collecting terms together

$$T_i^{j+1} = \sigma T_{i+1}^j - (2\sigma - 2\gamma + 1)T_i^j + \sigma T_{i-1}^j + \gamma T_i^{j+1} + \gamma T_i^{j-1}$$

b) When the conduction in the x-direction is neglected, the equation becomes

$$\frac{\partial T}{\partial x} = \frac{\alpha}{u}\left(\frac{\partial^2 T}{\partial y^2}\right)$$

Which is parabolic. This means it can be marched forward in x with time-marching being replaced by x-marching. The boundary conditions given are sufficient for this, i.e. an initial condition at x = 0, and values temperatures at the walls. If the conduction in the x direction is included, the equation now becomes elliptic. This means that the boundary conditions provided are not sufficient. This can also be seen from the update equation which can no longer be marched forward explicitly in the x direction. At the outlet, the update equation cannot work without some form of boundary condition, i.e. an outlet boundary condition, supplied. The problem is now a boundary value problem and it is only possible to get a solution by simultaneous computation across the entire domain.

ci)

$$T_i^{j+1} = \sigma T_{i+1}^j + (1-2\sigma)T_i^j + \sigma T_{i-1}^j$$

$$T_i^{j+1} = T' + \sigma(-1)^{i+1}\varepsilon + (1-2\sigma)(-1)^i\varepsilon + \sigma(-1)^{i-1}\varepsilon$$

$$T_i^{j+1} = T' + (-1)^i\,(-\sigma\varepsilon + (1-2\sigma)\varepsilon - \sigma\varepsilon)$$

$$T_i^{j+1} - T' = (-1)^i\varepsilon\,(1-4\sigma)$$

$$T_i^{j+1} - T' = \left(T_i^j - T'\right)(1-4\sigma)$$

So disturbance decay shrink if $|(1-4\sigma)| < 1$

$$-1 < (1-4\sigma) < 1$$

$$-2 < -4\sigma < 0$$

$$0 < \sigma < \frac{1}{2}$$

$$\frac{\alpha\Delta x}{u\Delta y^2} < \frac{1}{2}$$

Note that "otherwise" can note that when $(1-2\sigma)$ is negative, the new temperature is no longer a weighted average of the previous values at i, i+1 and i-1, so there is no smoothing filter.

cii)

$$T_i^{j+1} - T_i^j = \sigma T_{i+1}^{j+1} - (2\sigma)T_i^{j+1} + \sigma T_{i-1}^{j+1}$$

$$-T_i^j = \sigma T_{i+1}^{j+1} - (2\sigma+1)T_i^{j+1} + \sigma T_{i-1}^{j+1}$$

$$T_i^j = -\sigma T_{i+1}^{j+1} + (2\sigma+1)T_i^{j+1} - \sigma T_{i-1}^{j+1}$$

Now set the future j to have the disturbance and see how its magnitude has evolved.

$$T_i^{j+1} = T' + (-1)^i\varepsilon$$

$$T_i^j = T' + (-1)^i\varepsilon[4\sigma+1]$$

$$T_i^j = T' + (T_i^{j+1} - T')[4\sigma+1]$$

$$\frac{T_i^j - T'}{4\sigma+1} = (T_i^{j+1} - T')$$

This unstable if $|(1+4\sigma)| < 1$

$$-1 < 4\sigma + 1 < 1$$

$$-2 < 4\sigma < 0$$

$$-\frac{1}{2} < \sigma < 0$$

But since σ>0 then this implicit scheme is always stable. However, the implicit scheme cannot be marched explicitly, and instead a set of linear equations must be solved to move form j to j+1. So it's more work but more stable. However, if you make Δx too large, whilst it might be stable, the solution might be not very good. Though for large x it will converge correctly to the steady state.

7 (a) The central finite difference estimate for the first derivative is $\frac{\partial T}{\partial x} = \frac{T_{j+1}-T_{j-1}}{2\Delta x}$. Show that the leading order error term is $O(\Delta x^2)$?[25%]

$$y(x+\Delta x) = y_{j+1} = y(x) + \left.\frac{dy}{dx}\right|_x \Delta x + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_x \Delta x^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^3 + O(\Delta x^4)$$

$$y(x-\Delta x) = y_{j-1} = y(x) - \left.\frac{dy}{dx}\right|_x \Delta x + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_x \Delta x^2 - \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^3 + O(\Delta x^4)$$

Subtracting

$$y(x+\Delta x) = y_{j+1} - y_{j-1} = 2\left.\frac{dy}{dx}\right|_x \Delta x + \frac{2}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^3 + O(\Delta x^4)$$

So

$$\frac{y_{j+1} - y_{j-1}}{2\Delta x} = \left.\frac{dy}{dx}\right|_x + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^2 + O(\Delta x^4)$$

Therefore, the leading error term is $\frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^2$, which is $O(\Delta x^2)$ as required.

(b) Using three equally spaced grid points, find an expression for the highest order forward difference estimate of $\frac{\partial T}{\partial x}$.

$$y_{j+1} = y_i + \left.\frac{dy}{dx}\right|_x \Delta x + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_x \Delta x^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^3 + O(\Delta x^4)$$

$$y_{j+2} = y_i + \left.\frac{dy}{dx}\right|_x 2\Delta x + \frac{4}{2!}\left.\frac{d^2y}{dx^2}\right|_x \Delta x^2 + \frac{8}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^3 + O(\Delta x^4)$$

Eliminate the O(Δx^2)

$$y_{j+2} - 4y_{j+1} = -3y_i - 2\left.\frac{dy}{dx}\right|_x \Delta x + \frac{4}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^3 + O(\Delta x^4)$$

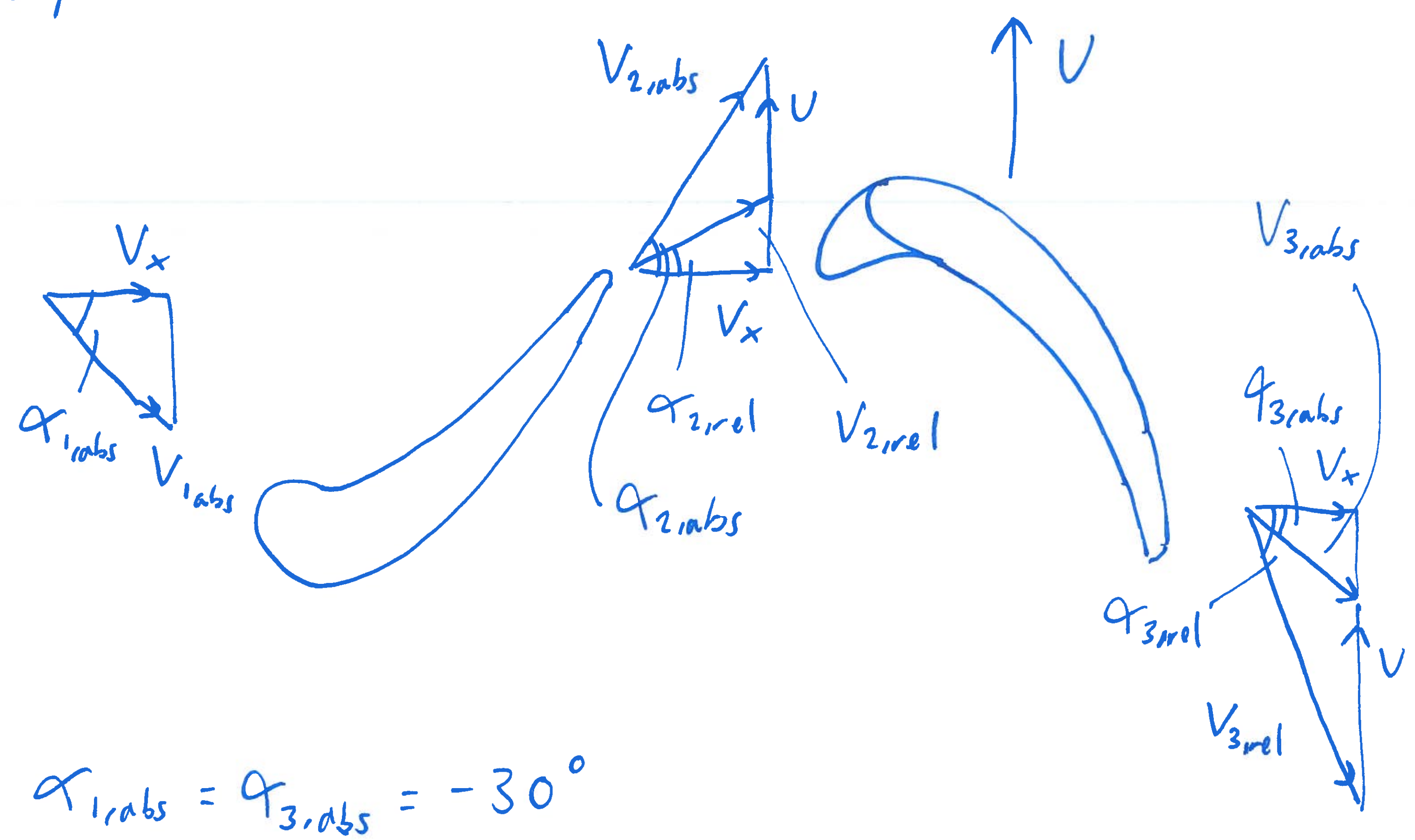
$$\frac{-y_{j+2}+4y_{j+1}-3y_i}{2\Delta x} = \left.\frac{dy}{dx}\right|_x + \frac{2}{3!}\left.\frac{d^3y}{dx^3}\right|_x \Delta x^2 + O(\Delta x^4)$$

Therefore

$$\left.\frac{dy}{dx}\right|_x = \frac{-y_{j+2} + 4y_{j+1} - 3y_i}{2\Delta x}$$

And the error is O(Δx^2)

a)



$\alpha_{1,abs} = \alpha_{3,abs} = -30^\circ$

$V_x = 200\,m/s$

$\phi = \dfrac{V_x}{U} = 0.5$

$C_p = 1150\,J/kg$

$\gamma = 1.333$

$\alpha_{2,rel} - \alpha_{3,rel} = 110^\circ$

$$V_{1,abs} = \frac{V_x}{\cos\alpha_{1,abs}} = \frac{200}{\cos(-30)} = 231\,ms^{-1}$$

$$U = \frac{V_x}{\phi} = 400\,ms^{-1}$$

$$V_x = V_{3,rel}\cos\alpha_{3,rel} = V_{2,rel}\cos\alpha_{2,rel} = 200 \qquad (2)$$

$$V_{3,rel}\sin\alpha_{3,rel} + U = V_{1,abs}\sin\alpha_{1,abs}$$

$$V_{3,rel}\sin\alpha_{3,rel} = -515.5\,ms^{-1} \qquad (1)$$

$$\frac{①}{②} = \tan\alpha_{3,rel} = \frac{-515.5}{200} \qquad \alpha_{3,rel} = -68.8°$$

$$\alpha_{2,rel} = 41.2°$$

$$V_{2,abs}\sin\alpha_{2,abs} = V_{2,rel}\sin\alpha_{2,rel} + U = 575\,ms^{-1}$$

$$V_{2,rel} = \frac{V_x}{\cos\alpha_{2,rel}} = 266\,ms^{-1}$$

$$\tan(\alpha_{2,abs}) = \frac{575}{200} \; \cancel{70.8°}$$

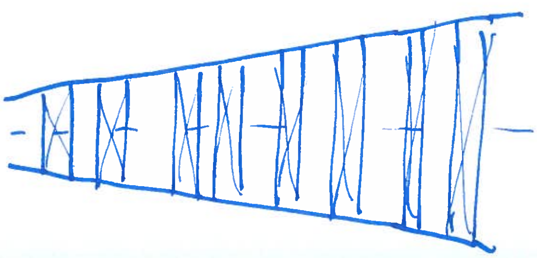
$$\alpha_{2,abs} = 70.8°$$

$$\Delta h_0 = U(V_{\theta 3} - V_{\theta 2})$$
$$= 400(231\sin(-30) - 575)$$
$$= -276\times10^3\ J\,kg^{-1}$$

For machine $4\times276\times10^3 = 1.10\ MJ\,kg^{-1}$

b) $V_{2,abs} = \dfrac{575}{\sin 70.8} = 610\ ms^{-1}$

$T_{01} = T_{02} = 2000K \qquad \dfrac{V_2}{\sqrt{c_pT_{02}}} = 0.402$



$M_2 = 0.73$ from tables

Span must increase as ρ falls to keep V_x. M increases as T_0 falls

a) i) $\Delta T_0 = \dfrac{\dot{W}_x}{\dot{m} c_p} = \dfrac{2000}{0.03 \times 1005} = 66.3\,K$

$T_{03} = T_{01} + \Delta T_0 = 354\,K$

$T_{03s} = T_0 \times PR^{\frac{\gamma-1}{\gamma}} = 335\,K$

$\eta_{TT} = \dfrac{T_{03s} - T_{01}}{T_{03} - T_{01}} = 71.1\%$

ii) $V_{r,3} = \dfrac{\dot{m}}{\rho_3 2\pi r_3 h} = \dfrac{0.03}{2\pi \times 34\times10^{-3} \times 3\times10^{-3}} = 46.8\,ms^{-1}$

$\rho_3 = 1\,kg\,m^{-3}$

$\Omega = \dfrac{2 \times \pi \times rpm}{60} = 9425\,rads^{-1}$

$V_{t,2} = \dfrac{\Delta T_0 \times c_p}{r_2 \Omega} = 295\,ms^{-1}$ No inlet swirl

$r_3 V_{t,3} = r_2 V_{t,2}$ No torque in diffuser

$V_{t,3} = 208\,ms^{-1}$

iii) $V_3 = \sqrt{V_{t,3}^2 + V_{r,3}^2} = 213\,ms^{-1}$

$\dfrac{V_3}{\sqrt{c_p T_{03}}} = 0.357$ $M_3 = 0.58$ tables

iv) $\dfrac{P_3}{P_{01}} = \dfrac{P_3}{P_{03}} \times \dfrac{P_{03}}{P_{01}} = 1.35$ $T_{3s} = T_{01}\left(\dfrac{P_3}{P_{01}}\right)^{\frac{\gamma-1}{\gamma}} = 314\,K$

$$\eta_{TS} = \frac{T_{3S} - T_{01}}{T_{03} - T_{01}} = 38.8\%$$

c) η_{TS} is correct metric. The compressor is drawing flow through the machine. The velocity at the exit, V_3, is wasted and KE is lost.

c) Speed increase:

$$\Omega_{new} = 12{,}566 \text{ rad s}^{-1} \qquad V_{t,2,new} = \frac{\Delta T_0 \times c_p}{r_2 \Omega_{new}}$$

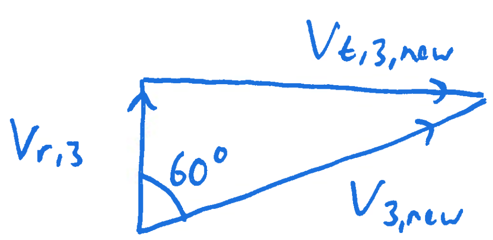
$$= 221 \text{ ms}^{-1}$$

$$V_{t,3,new} = \frac{r_2 V_{t,2,new}}{r_3} = 156 \text{ ms}^{-1}$$

$$V_{3,new} = \sqrt{V_{r,3}^2 + V_{t,3,new}^2} = 163 \text{ ms}^{-1}$$

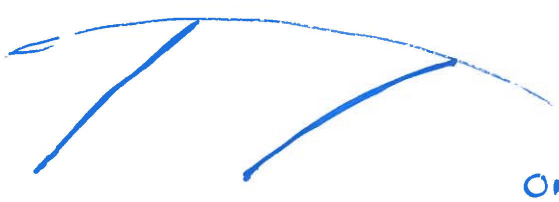
76% of old value

Diffuser vanes:



$$V_{3,new} = \frac{V_{r,3}}{\cos 60} = 93.6 \text{ ms}^{-1}$$

44% of old value



Vanes are superior.

Could cause extra cost through complexity.

or create noise with extra interactions

or cause problems for design with separation

or cause issues with [illegible]