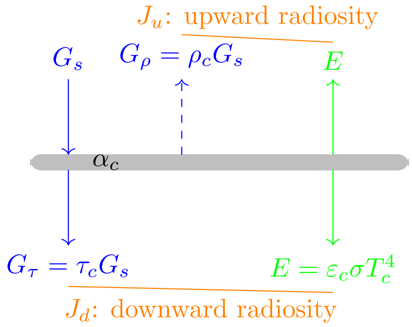
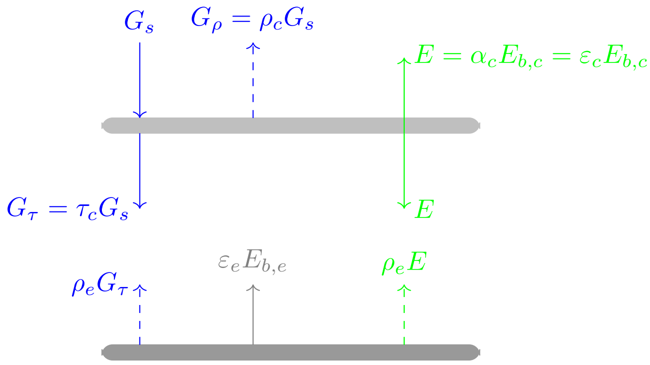
Module 3A6 - HEAT AND MASS TRANSFER - 2024
Solutions

1. Radiation with clouds

   (a) Diagram for cloud radiation



   (b) Diagram for cloud-earth interaction



$$\begin{aligned}
\text{Incoming} &: \tau_c G_s + \varepsilon_c E_{b,c} \\
\text{Outgoing} &: \rho_e(\tau_c G_s + \varepsilon_c E_{b,c}) + \varepsilon_e E_{b,e} = (1-\varepsilon_e)(\tau_c G_s + \varepsilon_c E_{b,c}) + \varepsilon_e E_{b,e} \\
\text{Net} &: \tau_c \varepsilon_e G_s + (1-\varepsilon_e)\varepsilon_c E_{b,c} - \varepsilon_e E_{b,e}
\end{aligned}$$

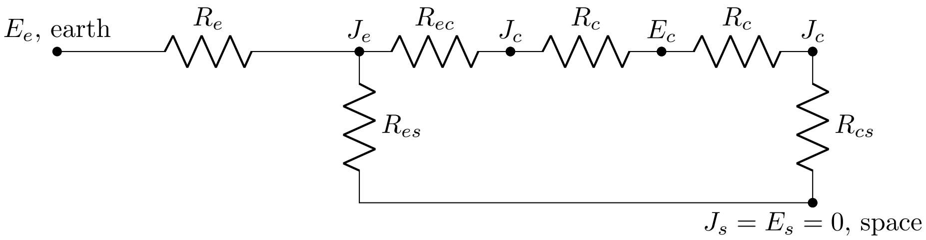
   (c) Net radiative flux to earth with cloud at T_c:

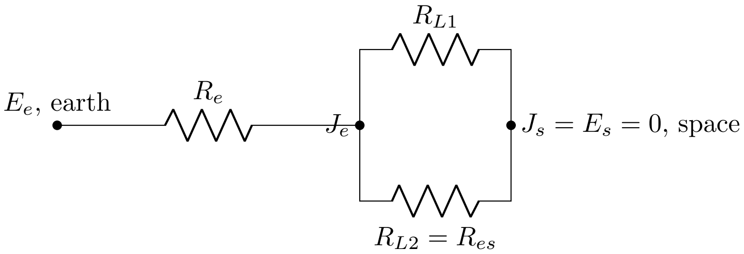
$$\begin{aligned}
q_{in} &= \tau_c G_s + \varepsilon_c E_{b,c} = (0.95)(1000)\text{W/m}^2 + (0.95)(5.67\times10^{-8}(\text{W/m}^2\text{K}^{-4})(240\text{ K})^4 \\
&= 950 + 179 = 1128\ \text{W/m}^2 \\
q_{out} &= \rho_e(\tau_c G_s + \varepsilon_c E_{b,c}) + \varepsilon_e E_{b,e} = (1-0.95)(1128\text{W/m}^2) + (0.95)(5.67\times10^{-8}(\text{W/m}^2\text{K}^{-4}) \\
&= 56 + 355 = 411\ \text{W/m}^2 \\
q_{net} &= 717\ \text{W/m}^2
\end{aligned}$$

   Without clouds:

$$q = G_s - \varepsilon_e E_e = (1000 - \underbrace{(0.95)(5.67\times10^{-8}(285\text{ K})^4)}_{355}\ \text{W/m}^2 = 645\ \text{W/m}^2$$

   (d) Equivalent circuit: earth can radiate either directly to space via the regions without cloud, or via the cloud as a barrier, where the resistance R_c appears both on the inner and outer sides:





Assume that the view factor of outer cloud to space and cloud to earth is $F_{cs} = F_{ce} = 1$. The view factor from earth to the area with no clouds F_{es} is therefore:

$$
\begin{aligned}
A_c F_{ce} &= A_e F_{ec} \\
F_{ec} &= \frac{A_c}{A_e} F_{ce} = \frac{1}{2} \\
F_{ec} + F_{es} &= 1 \\
F_{es} &= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
R_{es} &= \frac{1}{A_e F_{es}} = \frac{2}{A_e} \\
R_{cs} &= \frac{1}{A_c F_{cs}} = \frac{1}{A_c} = \frac{2}{A_e} \\
R_{ec} &= \frac{1}{A_e F_{ec}} = \frac{1}{\frac{1}{2} A_e} = \frac{2}{A_e} \\
R_e &= \frac{1-\varepsilon_e}{\varepsilon_e A_e} \\
R_c &= \frac{1-\varepsilon_c}{\varepsilon_c A_c} = \frac{1-\varepsilon_c}{\varepsilon_c} \frac{2}{A_e}
\end{aligned}
$$

We now calculate the total resistances between earth and cloud, and cloud and space.

$$
\begin{aligned}
R_{L1} = R_{ec} + 2R_c + R_{es} = \frac{2}{A_e} + 2\frac{1-\varepsilon_c}{\varepsilon_c}\frac{2}{A_e} &= 4.21\frac{1}{A_e} \\
R_{L2} = R_{cs} &= \frac{2}{A_e} \\
R_{||} = \frac{R_{L1}}{R_{L2}} R_{L1} + R_{L2} = \frac{4.21}{2} 4.21 + 2\frac{1}{A_e} = 1.35\frac{1}{A_e} R_{tot} = R_e + R_{||} &= 1.40\frac{1}{A_e}
\end{aligned}
$$

We can now calculate the overall heat transfer between E_c and E_s :

$$
q = \frac{Q}{A_e} = \frac{E_e - E_s}{R_{cs1}} = 5.67 \times 10^{-8} \frac{(285^4 - 0)}{1.40} = 265 \text{ W/m}^2
$$

2. Wire

   (a) A uniform radial temperature results if $\frac{\lambda_w}{D} \ll h$. We have:

$$\text{Nu} = \frac{hD}{\lambda_a} = (0.35 + 0.67Re^{0.52})Pr^{0.3}$$
$$\frac{\lambda_w}{\lambda_a} = \frac{100}{0.03} = 3 \times 10^4$$

   Reynolds number should be small for sufficiently small wire, so for

$$Re \approx 0$$
$$\text{Bi} = \frac{hD}{\lambda_w} = (0.35)Pr^{0.3}\frac{\lambda_a}{\lambda_w} \approx 10^{-4}$$

   Therefore, the heat transfer is limited by the heat transfer to air and the radial temperature should be uniform.

   (b) The balance for steady 1D heat conduction flux q_x per unit cross section $A = \pi D^2/4$ of wire along the wire, with convection q_c per unit external area of wire $P\ dx = \pi D\ dx$, is:

$$(q_x - q_{x+dx})A - q_c P\ dx + \dot{g}\ Adx = 0$$
$$-\lambda_w A\frac{dT}{dx} - \left(-\lambda_w A\frac{dT}{dx} - \lambda_w A\frac{d^2T}{dx^2}\ dx\right) - h(T - T_o)P\ dx = -\dot{g}\ A\ dx$$
$$\frac{d^2}{dx^2}(T - T_o) - \frac{hP}{\lambda_w A}(T - T_o) = -\frac{\dot{g}}{\lambda_w}$$
$$\frac{d^2\theta}{dx^2} - m^2\theta = B$$

   where $m^2 = \frac{hP}{\lambda_w A} = \frac{4h}{\lambda_w D}$, and $B = -\frac{\dot{g}}{\lambda_w}$.

   (c) The general solution can be shown to satisfy the homogeneous solution for the equation, as:

$$\frac{d^2\theta}{dx^2} = m^2(C_1 e^{mx} + C2e^{-mx}) = m^2\theta$$

   The particular solution $\theta = C_3$ satisfies the non-homogeneous equation:

$$(\frac{d^2\theta_h}{dx^2} - m^2\theta_h) - m^2 C_3 = B$$
$$C_3 = -\frac{B}{m^2}$$

   Applying boundary conditions:

$$\theta(+L) = C_1 \exp(mL) + C_2 \exp(-mL) - \frac{B}{m^2} = 0$$
$$\theta(-L) = C_1 \exp(-mL) + C_2 \exp(mL) - \frac{B}{m^2} = 0$$

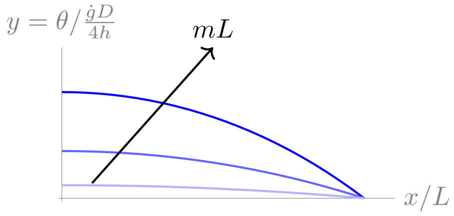
   which can be solved as:

$$C_1 = C_2 = \frac{B}{m^2}\frac{\exp(mL) - \exp(-mL)}{\exp(2mL) - \exp(-2mL)} = \frac{B}{m^2}\frac{1}{\exp(mL) + \exp(-mL)}$$

   so that:

$$\theta = \frac{B}{m^2}\left(\frac{\exp(mx) + \exp(-mx)}{\exp(mL) + \exp(-mL)} - 1\right) = \frac{\dot{g}D}{4h}\left(1 - \frac{\exp(mx) + \exp(-mx)}{\exp(mL) + \exp(-mL)}\right)$$

The solution for θ is linear in $\dot{g}$, so we just sketch the ratio $y = \theta / \frac{\dot{g}D}{4h}$ as a function of x/L and mL:



Acceptable answers could use arguments of symmetry and the operating boundary conditions for the sketch.

(d) The general equation contains the three terms (net conduction, net convection, heat generation) adding up to zero. Integrating over the domain once after multiplying by (λ_w), we have:

$$\lambda_w \frac{d^2}{dx^2}(T - T_o) - \frac{hP}{A}(T - T_o) = -\dot{g}$$

$$\underbrace{\lambda_w \frac{d\theta}{dx}(T - T_o)|_{-L}^{+L}}_{\text{net transfer at walls}} - \underbrace{\int_{-L}^{+L} \frac{hP}{A}(T - T_o)\, dx}_{\text{net convection}} + \underbrace{\int_{-L}^{+L} \dot{g}\, dx}_{\text{generation}} = 0$$

which can be integrated using the respective equations. However, by construction, this is not necessary, so long as the final equations are shown to satisfy the governing equations and respective boundary conditions. Solutions using θ as a solution and substituting to obtain the three terms are also acceptable, even though they take more effort.

3. (a) From the given values of F_m as a function of U_m, we have:

$$\begin{aligned}
F_{m,1} &= \frac{1}{2} C_{D,1} \rho U_1^2 A \\
F_{m,2} &= \frac{1}{2} C_{D,2} \rho U_2^2 A \\
\frac{F_{m,1}}{F_{m,2}} &= \frac{C_{D,1}}{C_{D2}} \frac{U_1^2}{U_2^2} = \frac{U_1^n}{U_2^n} \frac{U_1^2}{U_2^2} = \frac{U_1^{2+n}}{U_2^{2+n}} \\
8 &= 4^{2+n} \\
2^3 &= 2^{2(2+n)} \\
n &= -0.5
\end{aligned}$$

(b) From the Stanton number:

$$\begin{aligned}
\mathrm{St} &= \frac{h \Delta T}{\rho U c_p \Delta T} = \frac{hL}{\lambda} \frac{\lambda}{\rho c_p} \frac{1}{UL} = \mathrm{Nu}_L \frac{\alpha}{\nu} \frac{\nu}{UL} = \frac{\mathrm{Nu}_L}{\mathrm{Pr}\mathrm{Re}_L} \\
\mathrm{Nu}_L &= \mathrm{St}\mathrm{Pr}\mathrm{Re}_L = \frac{C_D}{2} \mathrm{Pr}\mathrm{Re}_L \propto \mathrm{Re}_L^{0.5}
\end{aligned}$$

(c) Considering the total force and ratio of forces, and using $r_U = U/U_m$ as the ratio of velocities,

$$\begin{aligned}
F &= \frac{1}{2} C_D \rho U^2 W H \\
\frac{F}{F_m} &= \frac{C_D}{C_{D,m}} r_U^2 r_L^2 = ((r_U)(r_L))^n \, r_U^2 r_L^2 = (r_U r_L)^{n+2} = (r_U r_L)^{1.5}
\end{aligned}$$

Therefore,

$$F = (7.7 \text{ kN}) (\frac{10}{8} 8)^{1.5} = 243 \text{ N}$$

For the heat transfer, we have:

$$\begin{aligned}
Q &= hA\Delta T \\
St &= \frac{h}{\rho c_p U} = \frac{C_D}{2} = \frac{F}{\rho U^2 A} \\
h &= \frac{F}{\rho U^2 A} \rho c_p U = \frac{F}{UA} c_p \\
Q &= hA\Delta T = \frac{F}{U} c_p \Delta T
\end{aligned}$$

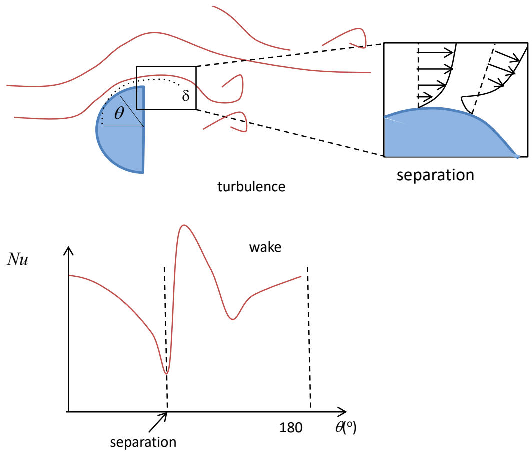
We can use the result above for F scaling with r_L and r_U as:

$$\begin{aligned}
Q = \frac{F}{U} c_p \Delta T &= \frac{F}{F_m} \frac{F_m}{U_m} \frac{U_m}{U} c_p \Delta T \\
&= (r_U r_L)^{n+2} \frac{F_m}{U_m} r_U^{-1} c_p \Delta T
\end{aligned}$$

From the databook, $c_p = 1.005$ kg/kJ K.

$$Q = r_U^{n+1} r_L^{n+2} \frac{F_m}{U_m} c_p \Delta T = \left(\frac{10}{8}\right)^{0.5} 8^{1.5} \frac{7.7 \text{ N}}{8 \text{ m s}^{-1}} (1005 \text{ J kg}^{-1} \text{ K}^{-1})(15 \text{ K}) = 364 \text{ kW}$$

NB: For some unknown reason, reason the value of the model length scale was not given in the original question statement, so all entries that include r_L were considered acceptable, as above, or if a certain value was assumed.



(d) The maximum heat transfer rate is at the stagnation point (thinnest boundary layer), decreasing up to the ledge of the bluff body. The heat transfer increases in the wake, due to turbulent motion. The sketch should include both maximum at the stagnation point, a decrease (possibly a separation point), followed by an increase and approximately uniform heat transfer rate in the back end. The increase in velocity should lead to an increase in heat transfer rate and onset of turbulence with higher heat transfer.

(e) The density and viscosity of the fluid would change, and so would the relevant Prandtl and Reynolds number. In addition, the range of Reynolds number could be very different from the regime discussed, so that the relationships used for the drag coefficient might not be valid.

4. Droplet

(a) Using a spherical coordinate system in the frame of reference of the droplet, and observing that $\boldsymbol{u} \cdot \nabla(\rho Y) = \nabla \cdot (\rho \mathbf{u} Y)$ for uniform density steady flows, we have for the balance of water vapour:

$$\nabla \cdot (\rho \mathbf{u} Y - D \nabla \rho Y) = 0$$
$$\frac{1}{r^2} \frac{d}{dr} r^2 \left(\rho u Y - \rho D \frac{dY}{dr} \right) = 0$$

Integrating,

$$r^2 (\rho u Y - \rho D \frac{dY}{dr}) = \frac{\dot{m}}{4\pi}$$

where the latter equality appears because we assume a quasi-steady state, so that the total mass evaporation at the surface of the droplet must be equal to the total mass flow rate at all radii, so that $\dot{m} = 4\pi \rho u r^2$. We now use the fact that $\rho u = \frac{\dot{m}}{4\pi r^2}$ and rearrange the terms, so that:

$$-\rho D \frac{dY}{dr} = \frac{\dot{m}}{4\pi r^2}(1 - Y)$$
$$\frac{d \ln(1 - Y)}{dr} = \frac{\dot{m}}{4\pi \rho D} \frac{1}{r^2}$$

Integrating, and using the boundary condition at $r \to \infty$, we have

$$\ln(1 - Y) = -\frac{\dot{m}}{4\pi \rho D} \frac{1}{r} + \ln(1 - Y_\infty)$$
$$\ln \left(\frac{1 - Y}{1 - Y_\infty} \right) = -\frac{\dot{m}}{4\pi \rho D} \frac{1}{r}$$

Therefore, at the surface $r = R$ we now have:

$$\frac{\dot{m}}{4\pi \rho D} \frac{1}{R} = \ln \left(\frac{1 - Y_\infty}{1 - Y_s} \right)$$

q.e.d.

(b) The mass of the droplet is $m = \rho_l \frac{4\pi}{3} R^3$, so that the rate of change must be equal to the mass leaving or condensing on the the droplet:

$$\rho_l 4\pi R^2 \frac{dR}{dt} = -\dot{m} = -4\pi \rho D R \ln \left(\frac{1 - Y_\infty}{1 - Y_s} \right)$$
$$\rho_l R \frac{dR}{dt} = -\rho D \ln \left(\frac{1 - Y_\infty}{1 - Y_s} \right)$$
$$\frac{dR^2}{dt} = -2D \; \frac{\rho}{\rho_l} \ln \left(\frac{1 - Y_\infty}{1 - Y_s} \right)$$

which is negative (evaporation) if $Y_\infty < Y_s$, or positive (condensation) otherwise. The value is constant if Y_s is constant, i.e. if T_s is constant.

(c) The temperature affects the vapour pressure of water at the surface, so the surface concentration varies accordingly:

$$X_s - X_\infty = (Y_s - Y_\infty)\frac{W_a}{W_w} = \frac{p_s - p_{s,\infty}}{p_a} = K\frac{T_s - T_\infty}{T_\infty}$$
$$(Y_s - Y_\infty) = K\frac{W_w}{W_a}K\frac{T_s - T_\infty}{T_\infty}$$

From the given temperature evolution:

$$T_s - T_{s,0} = (T_\infty - T_{s,0})(1 - e^{-t/\tau})$$
$$T_s - T_\infty = -(T_\infty - T_{s,0})e^{-t/\tau}$$
$$Y_s = Y_\infty - K\frac{W_w}{W_a}K\frac{(T_\infty - T_{s,0})}{T_\infty}e^{-t/\tau}$$
$$\frac{dR^2}{dt} = -2D\ \frac{\rho}{\rho_l}\ln\left(\frac{1-Y_\infty}{1-Y_s}\right) = -2D\ \frac{\rho}{\rho_l}\ln\left(1 - \frac{K\frac{W_w}{W_a}K\frac{(T_\infty - T_{s,0})}{T_\infty}e^{-t/\tau}}{1-Y_\infty}\right)$$

The rate of growth of R^2 positive (condensation), and tends to zero as the temperature of the droplet increases towards T_∞, and the rate of radius growth goes to zero.