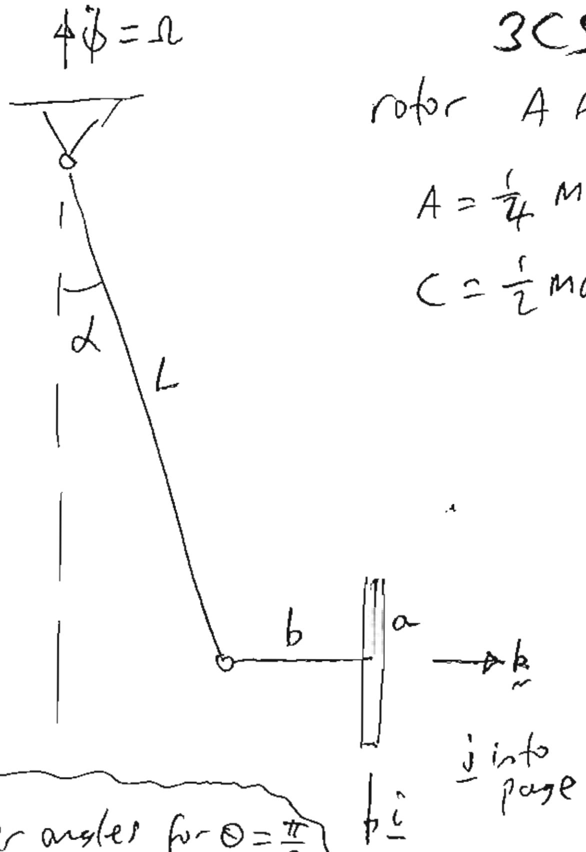
3C5 Q1

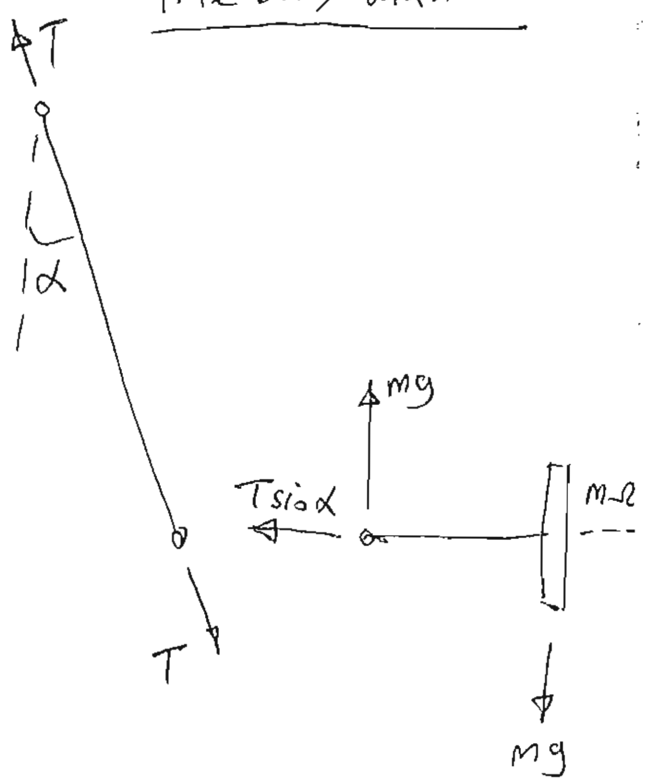
rotor AAC

$A = \frac{1}{4} m a^2$

$C = \frac{1}{2} m a^2$

Free body diagrams





Euler angles for $\theta = \frac{\pi}{2}$

$\Omega_1 = -\dot{\phi}\sin\theta = -\Omega$

$\Omega_2 = \dot{\theta} = 0$

$\Omega_3 = \dot{\phi}\cos\theta = 0$

$T\cos\alpha = mg$

$T\sin\alpha = m\Omega^2 R$

$\therefore \; mg\tan\alpha = m\Omega^2(b + L\sin\alpha)$

$\therefore \; g\tan\alpha = \Omega^2(b + L\sin\alpha)$

Assume small α $\therefore$ $\boxed{\Omega^2 \simeq \frac{g\alpha}{b}}$ ①

Gyro ② $\therefore$ $Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$

$\therefore \; mgb = 0 + (0 - C\omega_3)\Omega_1 = \frac{1}{2} m a^2 \omega \Omega$

$\therefore \; \boxed{\Omega = \frac{2gb}{a^2\omega}}$ ②

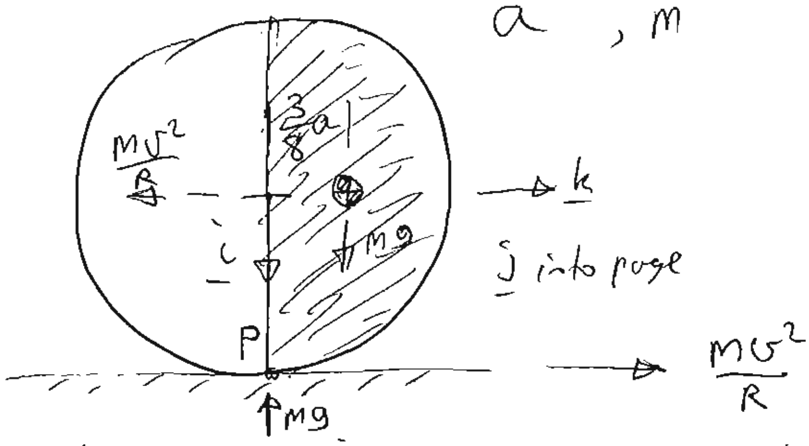
① & ② $\therefore$ $\frac{g\alpha}{b} \simeq \frac{4g^2b^2}{a^4\omega^2}$ $\therefore$ $\boxed{\alpha \simeq \frac{4gb^3}{a^4\omega^2}}$

Check small angle assumption is consistent with fast spin,

yes $\omega \to \infty$ $\therefore$ $\alpha \to 0$

3C5 Q2

a , m $C = \frac{2}{5} m a^2$



$R \gg a$

$v = \Omega_1 R$

$\Omega_1 = -\dot{\phi} \sin\theta$
$\Omega_2 = \dot{\theta}$
$\Omega_3 = \dot{\phi} \cos\theta$

$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$

no slip $\underline{v}_P = \underline{\omega} \times a\underline{i} + v\underline{j} = 0$

$\therefore \; -a\omega_2 \underline{k} + a\omega_3 \underline{j} = -v\underline{j}$

$\therefore \; \omega_2 = 0 \qquad \omega_3 = -\frac{v}{a}$

Gyro eq (2) in steady state

$Q_2 = (A\Omega_3 - C\omega_3)\Omega_1$ fast spin $\Omega_3 \ll \omega_3$

$\therefore \; mg\frac{3}{8}a - \frac{mv^2}{R}a = -\frac{2}{5}ma^2\left(\frac{-v}{a}\right)\cdot\frac{v}{R}$

$\therefore \; g\frac{3}{8} = \left(\frac{2}{5} + 1\right)\frac{v^2}{R}$

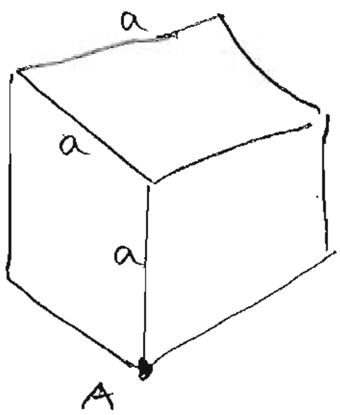
$\therefore \; R = \frac{7}{5}\frac{8}{3}\frac{v^2}{g} = \frac{56}{15}\frac{v^2}{g}$

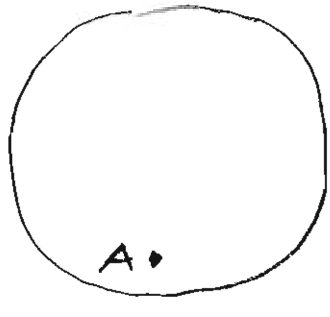
friction $\frac{mv^2}{R\,mg} \leq \mu \quad \therefore \frac{v^2}{Rg} \leq \mu \quad \therefore \mu \geq \frac{15}{56}$

(a) (i) The cube has principal moments of inertia

$\frac{1}{6}ma^2$, $\frac{1}{6}ma^2$, $\frac{1}{6}ma^2$ at its centre

(straight from data book)





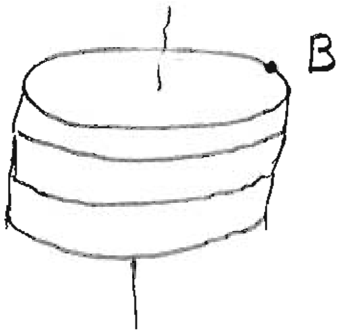
It is like a sphere here hence the point A is defined only by its distance $= a\sqrt{3}/2$ from the centre

Hence the principal moments of inertia at A are $\frac{1}{6}ma^2$, $\frac{11}{12}ma^2$, $\frac{11}{12}ma^2$

(using parallel axis theorem $\frac{1}{6}ma^2 + m\left(\frac{\sqrt{3}a}{2}\right)^2$

$= \frac{11}{12}ma^2$)

(ii) The cube can be visualised as three equal discs, stacked



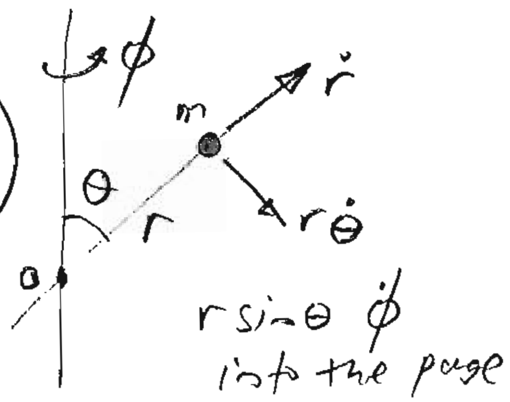
The top disc (note that a square plate is "AAC" just like a circular disc) can be rotated without changing any inertial properties hence the principal moments of inertia at B are $\frac{1}{6}ma^2$, $\frac{11}{12}ma^2$, $\frac{11}{12}ma^2$ as before

(b) (i) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

if these are zero

then $\dfrac{\partial T}{\partial \dot{q}_i} = \text{const}$ is a conservation law

(ii) $$T = \tfrac{1}{2} m v^2$$
$$= \tfrac{1}{2} m\left(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\phi})^2\right)$$



<u>VELOCITIES</u>

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial T}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$$\frac{\partial T}{\partial r} = m\dot{\theta}^2 r + m\sin^2\theta\,\dot{\phi}^2 r \qquad \frac{\partial T}{\partial \theta} = mr^2\dot{\phi}^2\sin\theta\cos\theta$$

$$\frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial V}{\partial r} = \frac{GMm}{r^2} - \frac{3J_2a^2GMm}{2r^4}(3\cos^2\theta - 1)$$

$$\frac{\partial V}{\partial \theta} = -\frac{GMmJ_2a^2}{r^3}\,3\cos\theta\sin\theta \qquad \frac{\partial V}{\partial \phi} = 0$$

Conservation law from ϕ

$$\therefore\ r^2\sin^2\theta\,\dot{\phi} = \text{const}$$

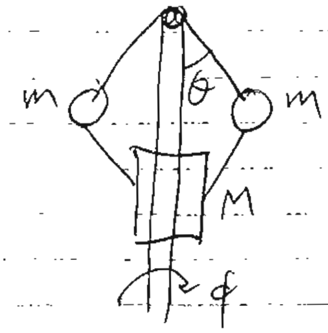
The other equations are

$$r: \quad \ddot{r} - \dot{\theta}^2 r + \sin^2\theta\, \dot{\phi}^2 r + \frac{GM}{r^2} - \frac{3GMJ_2a^2}{2r^4}\left(3\cos^2\theta - 1\right) = 0$$

$$\theta: \quad r^2\ddot{\theta} - r^2\dot{\phi}^2 \sin\theta\cos\theta + \frac{3GMJ_2a^2}{r^3}\cos\theta\sin\theta =$$

4(a) Use θ, ϕ as generalised coordinates, where $\dot\phi = \Omega$

Masses m have velocity $a\dot\theta$ in the plane of the diagram and $a\sin\theta\,\dot\phi$ into the page. Mass M has velocity $2a\sin\theta\,\dot\theta$ vertically.



So kinetic energy

$$T = 2\left\{\tfrac{1}{2}m\left(a^2\dot\theta^2 + a^2\sin^2\theta\,\dot\phi^2\right)\right\} + \tfrac{1}{2}M\left(2a\sin\theta\,\dot\theta\right)^2$$

Potential energy $V = 2mga(1-\cos\theta) + M.2ga(1-\cos\theta)$

Lagrange equations

$$\phi: \quad \frac{d}{dt}\left(2ma^2\sin^2\theta\,\dot\phi\right) = Q = \text{driving torque on shaft}$$

$$\theta: \quad \frac{d}{dt}\left[a^2\left(m+2M\sin^2\theta\right).2\dot\theta\right] - 4Ma^2\sin\theta\cos\theta\,\dot\theta^2 - 2ma^2\sin\theta\cos\theta\,\dot\phi^2 + 2(m+M)ga\sin\theta = 0$$

(b) Put $\dot\phi = \Omega = $ constant and simplify:

$$\left(m+2M\sin^2\theta\right)\ddot\theta + 2M\sin\theta\cos\theta\,\dot\theta^2 - m\sin\theta\cos\theta\,\Omega^2 + (m+M)\frac{g}{a}\sin\theta = 0 \qquad (1)$$

(c) If $\theta = $ constant, require $m\sin\theta\cos\theta\,\Omega^2 = (m+M)\frac{g}{a}\sin\theta$

so <u>either</u> $\sin\theta = 0$, so $\theta = 0$ or π (impossible)

<u>or</u> $\cos\theta = \dfrac{(m+M)g}{ma\Omega^2}$, OK provided ≤ 1

<u>Case 1</u>: $\Omega^2 < (m+M)g/ma$: only root is $\theta = 0$

So perturb: let θ be small, and linearise (1)

Then $m\ddot\theta - m\theta\Omega^2 + (m+M)\frac{g}{a}\theta \simeq 0$

ie $\ddot\theta + \left[\dfrac{(m+M)g}{ma} - \Omega^2\right]\theta \simeq 0$

4 (c) contd: So $\theta = 0$ solution is stable and the motion for small θ is oscillatory with angular frequency

$$\omega = \sqrt{\frac{(m+M)g}{ma} - \Omega^2}$$

Case 2: $\Omega^2 > \frac{(m+M)g}{ma}$

Previous analysis shows that $\theta = 0$ solution is now unstable.

Investigate the other solution: let $\theta = \alpha + \delta$ where $\cos\alpha = \frac{(m+M)g}{ma\Omega^2}$ and $|\delta| \ll 1$

Linearise (1) again:

$$(m + 2M\sin^2\alpha)\ddot{\delta} - m(\sin\alpha + \delta\cos\alpha)(\cos\alpha - \delta\sin\alpha)\Omega^2 + (m+M)\frac{g}{a}(\sin\alpha + \delta\cos\alpha) \simeq 0$$

$$\therefore (m + 2M\sin^2\alpha)\ddot{\delta} + \left\{m\Omega^2\sin^2\alpha - m\Omega^2\cos^2\alpha + (m+M)\frac{g}{a}\cos\alpha\right\}\delta \simeq 0$$

$$\therefore (m + 2M\sin^2\alpha)\ddot{\delta} + m\Omega^2\sin^2\alpha\,\delta \simeq 0$$

So this solution is always stable when it exists, and the frequency of small oscillations in δ is

$$w = \sqrt{\frac{m\Omega^2\sin^2\alpha}{(m + 2M\sin^2\alpha)}}$$