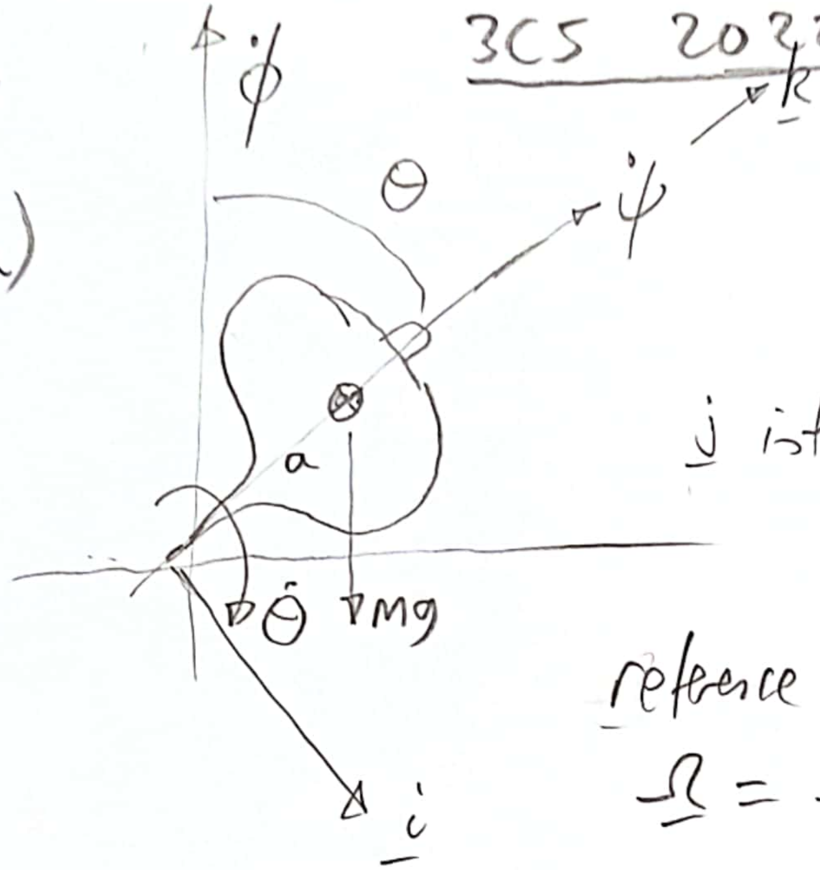
① 3C5 2022 crib

(a)



$Q_2 = mga \sin\theta$

$\underline{j}$ into page

reference frame

$\underline{\Omega} = \Omega_1 \underline{i} + \Omega_2 \underline{j} + \Omega_3 \underline{k}$

body

$\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 k$

$\underline{\omega} = \underline{\Omega} + \dot{\psi} \underline{k}$ (the body spins $\dot{\psi} \underline{k}$ relative to the reference frame)

$\therefore \omega_1 = \Omega_1 \qquad \omega_2 = \Omega_2$

$\omega_3 = \Omega_3 + \dot{\psi} \qquad \therefore \omega_3 - \Omega_3 = \dot{\psi}$

resolving
$$\Omega_1 = -\dot{\phi} \sin\theta \qquad = \omega_1$$
$$\Omega_2 = \dot{\theta} \qquad = \omega_2$$
$$\Omega_3 = \dot{\phi} \cos\theta$$

(b) Gyro equation (2)

$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$

Steady state $\dot{\Omega}_2 = 0$

$\therefore \quad mga\sin\theta = -(A\dot{\phi}\cos\theta - C\omega_3)\dot{\phi}\sin\theta$

$\therefore \quad \sin\theta = 0$

or $\quad A\cos\theta\,\dot{\phi}^2 - C\omega_3\dot{\phi} + mga = 0 \qquad (1)$

Fast spin and slow precession $\therefore$ ignore $\dot{\phi}^2$

$\therefore \quad \boxed{\dot{\phi} = \dfrac{mga}{C\omega_3}}$

(c) Nutation (which can be superposed on precession)

occurs even if $mg = 0$

$\therefore \quad A\cos\theta\,\dot{\phi} - C\omega_3 = 0$

$\therefore \quad \boxed{\dot{\phi} = \dfrac{C\omega_3}{A}} \quad =$ nutation frequency

small oscillations

$\theta \sim 0$

$\cos\theta \sim 1$

(d)

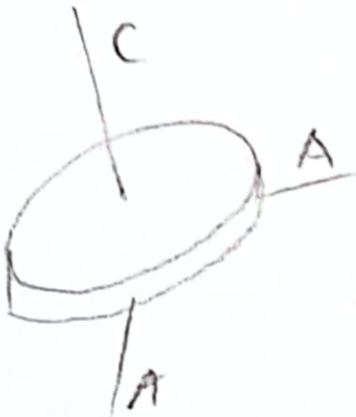
(1) only has a real solution if

$(C\omega_3)^2 - 4A\cos\theta\, mga > 0$

$\therefore \quad \boxed{\omega_3^2 > \dfrac{4Amga}{C^2}} \quad$ for $\theta \sim 0$

(2)

(a)



$$A\dot{\omega}_1 - (\overset{A}{\cancel{B}} - C)\omega_2\omega_3 = 0$$
$$\overset{A}{\cancel{B}}\dot{\omega}_2 - (C-A)\omega_3\omega_1 = 0$$
$$C\dot{\omega}_3 - (A - \underset{A}{\cancel{B}})\omega_1\omega_2 = 0$$

Disc
$C = 2A$

$$\therefore \quad A\dot{\omega}_1 - (A-C)\omega_2\omega_3 = 0$$
$$A\dot{\omega}_2 + (A-C)\omega_3\omega_1 = 0$$
$$C\dot{\omega}_3 = 0$$

(i) $\therefore \ \omega_3 = \text{const}$

$$\dot{\omega}_1 + \omega_3\omega_2 = 0$$
$$\dot{\omega}_2 - \omega_3\omega_1 = 0$$

(ii)

$$\therefore \ \ddot{\omega}_1 + \omega_3\dot{\omega}_2 = 0$$

(b) $\therefore \ \boxed{\ddot{\omega}_1 + \omega_3^2\,\omega_1 = 0}$ SHM at frequency ω_3 ✓

(c) Gyro equation (2) & (3)

$\Omega_1 = -\dot{\phi}\sin\theta$
$\Omega_2 = \dot{\theta}$
$\Omega_3 = \dot{\phi}\cos\theta$

$$0 = Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$
$$0 = Q_3 = C\dot{\omega}_3$$

$\therefore \ \omega_3 = \text{const}$

$Q_2 = 0$, steady state nutation (wobbling)

$$\dot{\theta} = \text{const} \quad \therefore \dot{\Omega}_2 = 0$$

$$\therefore \left(A\dot{\phi}\cos\theta - C\omega_3\right)\dot{\phi}\sin\theta = 0$$

small θ and $\dot{\phi} \neq 0$

$$\therefore A\dot{\phi} = C\omega_3$$

$$\therefore \dot{\phi} = \frac{C}{A}\omega_3 = 2\omega_3 \text{ (thin disc)}$$

Not ω_3 as before because the earlier answer was in a body-fixed reference frame and the body is spinning at ω_3 so the wobbling rate observed in a space-fixed frame is the sum of the body fixed rate + the spin rate

$$\omega_3 + \omega_3 = 2\omega_3 \quad ✓$$

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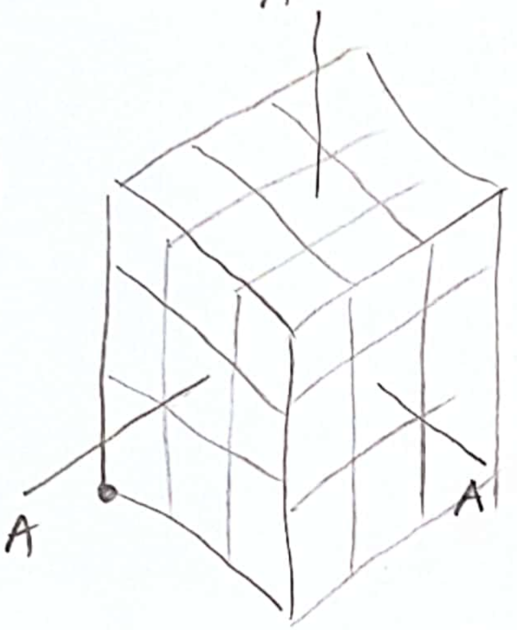
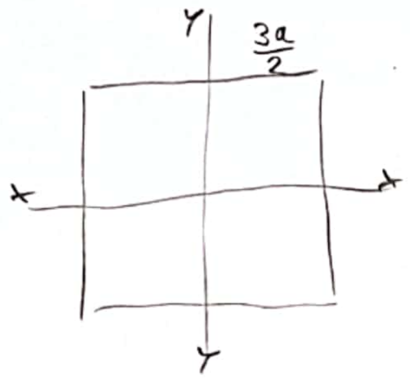
3 (a)

(i)



Solid cube
Mass 27m
side 3a

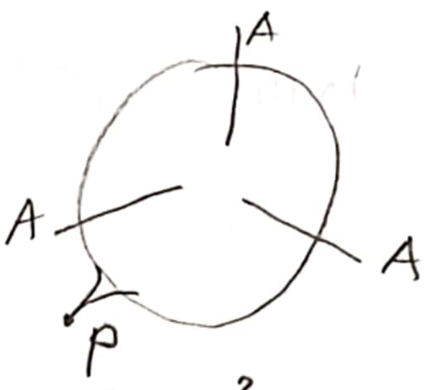
"AAA"



$$I_{xx} = I_{yy}$$
$$= \frac{1}{3} 27m \left(\frac{3a}{2}\right)^2$$
$$= \frac{81ma^2}{4}$$

$$A = I_{xx} + I_{yy} = \frac{81ma^2}{2} \quad //$$

As an AAA body, it's the same as a sphere so at P it's an "AAC" body with $C = A = \frac{81ma^2}{2}$



and the new $A = \frac{81ma^2}{2} + 27m\left(\frac{3a}{2}\sqrt{3}\right)^2$

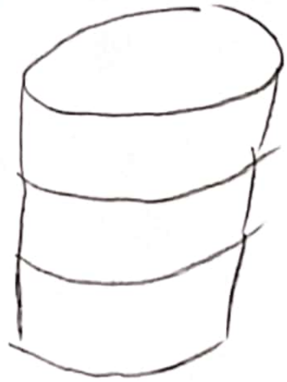
$$= \frac{81}{2}\left(1 + \frac{9}{2}\right)ma^2 = \frac{891}{4}ma^2$$

So at P we have

$$\frac{81 ma^2}{2} \qquad \frac{81 ma^2}{2} \qquad \frac{891 ma^2}{4}$$

(ii) The Cube is identical to a stack of three discs

(a square plate is the same as a circular disc).



Rotating the discs makes no difference

$\therefore$ at Q the values are unchanged

$$\frac{81 ma^2}{2} \qquad \frac{81 ma^2}{2} \qquad \frac{891 ma^2}{4}$$

(b) $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M\dot{y}^2 = \frac{1}{2}\frac{P_x^2}{M} + \frac{1}{2}\frac{P_y^2}{M}$

(i) $P_x = M\dot{x}$ $\quad P_y = M\dot{y}$ $\qquad V = Mgy$

(ii) $H = P_x\dot{x} + P_y\dot{y} - \frac{1}{2}\frac{P_x^2}{M} - \frac{1}{2}\frac{P_y^2}{M} + Mgy$

$= \frac{P_x^2}{M} + \frac{P_y^2}{M} - \frac{1}{2}\frac{P_x^2}{M} - \frac{1}{2}\frac{P_y^2}{M} + Mgy$

$= \frac{1}{2}\frac{P_x^2}{M} + \frac{1}{2}\frac{P_y^2}{M} + Mgy$

(iii) $F = \dot{x}^2(\dot{y}^2 + 2gy)$

$= \left(\frac{P_x}{M}\right)^2\left(\left(\frac{P_y}{M}\right)^2 + 2gy\right)$

$\frac{dF}{dt} = 0 + \{F, H\} = \cancel{\frac{\partial F}{\partial x}\frac{\partial H}{\partial P_x}} - \cancel{\frac{\partial F}{\partial P_x}\frac{\partial H}{\partial x}} + \frac{\partial F}{\partial y}\frac{\partial H}{\partial P_y} - \frac{\partial F}{\partial P_y}\frac{\partial H}{\partial y}$

(iv) $= \cancel{-\frac{2P_x}{M^2}\left(\left(\frac{P_y}{M}\right)^2 + 2gy\right)} \; \left(\frac{P_x}{M}\right)^2 2g\frac{P_y}{M} - \left(\frac{P_x}{M}\right)^2\frac{2P_y}{M^2}Mg$

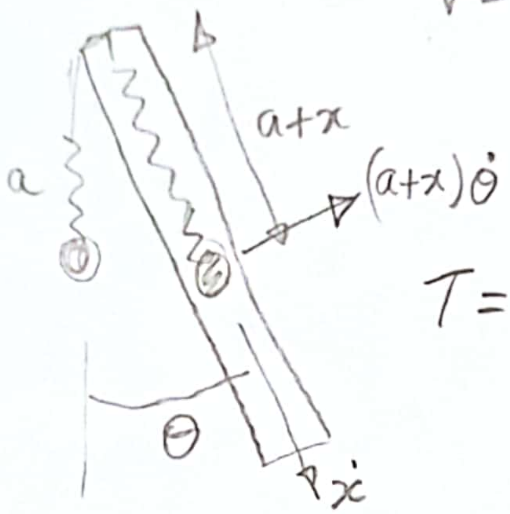
$= 0$ ✓

$M\dot{x}$ is constant (momentum)

$\frac{1}{2}M\dot{y}^2 + Mgy$ is const (energy)

so the product is conserved.

④ (a)



$$V = -Mg(a+x)\cos\theta + \frac{1}{2}kx^2 - \rho L g \frac{L}{2}\cos\theta$$

$$T = \frac{1}{2}M\dot x^2 + \frac{1}{2}M\left((a+x)\dot\theta\right)^2 + \frac{1}{2}\frac{1}{3}\rho L^3\dot\theta^2$$

$$p_x = \frac{\partial T}{\partial \dot x} = M\dot x$$

$$p_\theta = \frac{\partial T}{\partial \dot\theta} = \left(M(a+x)^2 + \frac{1}{3}\rho L^3\right)\dot\theta$$

(b)

$$\frac{\partial T}{\partial x} = M(a+x)\dot\theta^2 \qquad \frac{\partial V}{\partial x} = -Mg\cos\theta + kx$$

$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial V}{\partial \theta} = Mg(a+x)\sin\theta + \rho\frac{L^2 g}{2}\sin\theta$$

$$\therefore \quad M\ddot x - M(a+x)\dot\theta^2 + kx - Mg\cos\theta = 0$$

$$\left(M(a+x)^2 + \frac{1}{3}\rho L^3\right)\ddot\theta + 2M(a+x)\dot x\dot\theta + Mg(a+x)\sin\theta + \frac{1}{2}\rho L^2 g\sin\theta = 0$$

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(c)

(i) $M = x = 0$

$$I_A = \frac{1}{3}\rho L^3$$

$$\frac{1}{3}\rho L^3 \ddot{\theta} + \rho L g \frac{L}{2}\sin\theta = 0$$

$$\therefore \ddot{\theta} + \frac{3}{2}\frac{g}{L}\sin\theta = 0 \quad ✓$$

(ii) $\rho = x = 0$

$$Ma^2\ddot{\theta} + Mga\sin\theta = 0 \quad ✓$$

(iii) $\theta = 0$

$$M\ddot{x} + kx = Mg \quad ✓$$

(iv) $g = a = k = \rho = 0$

$$M\ddot{x} - x\dot{\theta}^2 = 0 \quad ✓$$

$$Mx^2\ddot{\theta} + 2Mx\dot{x}\dot{\theta} = 0$$

$$\therefore x^2\ddot{\theta} + 2x\dot{x}\dot{\theta} = 0$$

$$\therefore \frac{d}{dt}\left(x^2\dot{\theta}\right) = 0 \quad ✓$$

$\rho = 0$

$$M\ddot{x} + kx = Mg$$

$$Ma^2\ddot{\theta} + Mga\theta = 0$$

$$\therefore \ddot{x} + \frac{k}{M}x = g$$

$$\ddot{\theta} + \frac{g}{a}\theta = 0$$