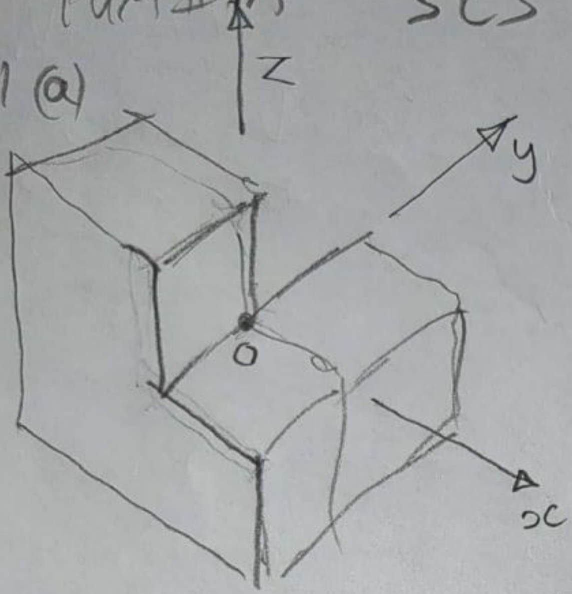
Part II A 3C5 2023 crib

Corresponds to Version HEMH/5

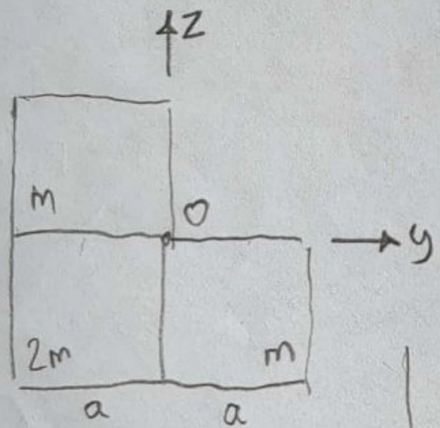
Q1 (a)



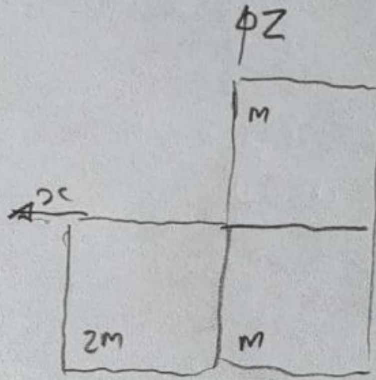
Piece 1

$$[I_1] = \frac{ma^2}{6}\begin{bmatrix} 16 & -3 & 3 \\ -3 & 16 & 0 \\ 3 & 0 & 16 \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

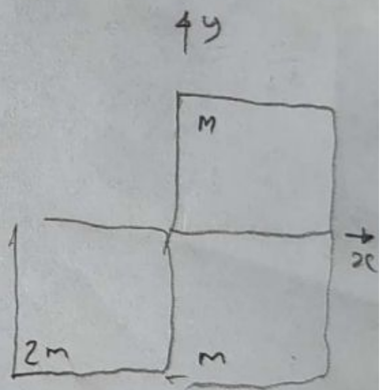
view x



view y



view z



$$I_{xx} = 4m\,\frac{1}{6}ma^2 + 4m\left(\frac{a}{\sqrt{2}}\right)^2$$
$$= \frac{2ma^2}{3} + 2ma^2$$
$$= \frac{8}{3}ma^2$$

$$I_{yz} = m\left(\frac{a}{2}\right)\left(-\frac{a}{2}\right) + m\left(-\frac{a}{2}\right)\left(\frac{a}{2}\right) + 2m\left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)$$
$$= 0$$

$$I_{yy} = I_{xx} = \frac{8}{3}ma^2$$

$$I_{xz} = 3m\left(\frac{a}{2}\right)\left(-\frac{a}{2}\right) + m\left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)$$
$$= -\frac{ma^2}{2}$$

$$I_{zz} = I_{xx} = \frac{8}{3}ma^2$$

$$I_{xy} = m\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + m\left(\frac{a}{2}\right)\left(-\frac{a}{2}\right) + 2m\left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)$$
$$= +\frac{ma^2}{2}$$

Q1(b)

Piece 2 is Piece 1 rotated by 180° about y axis so change signs of x & z so that the only elements with a sign change are $\int xy\,dm$ and $\int yz\,dm$

so $$[I_2] = \frac{ma^2}{6}\begin{bmatrix} 16 & 3 & -3 \\ 3 & 16 & 0 \\ -3 & 0 & 16 \end{bmatrix}$$

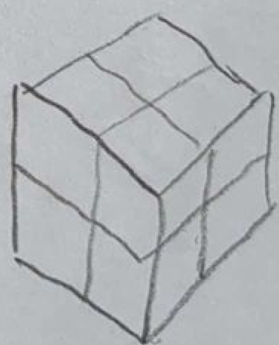
(c)

Add the two pieces together

$$[I_1]+[I_2] = \frac{ma^2}{6}\begin{bmatrix} 16 & -3 & 3 \\ -3 & 16 & 0 \\ 3 & 0 & 16 \end{bmatrix} + \frac{ma^2}{6}\begin{bmatrix} 16 & 3 & -3 \\ 3 & 16 & 0 \\ -3 & 0 & 16 \end{bmatrix}$$

$$= \frac{ma^2}{6}\begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$$

The expected value of A for a cube of mass 8m and side 2a is $\frac{1}{6} 8m(2a)^2 = \frac{32ma^2}{6}$ ✓ $= \frac{16ma^2}{3}$

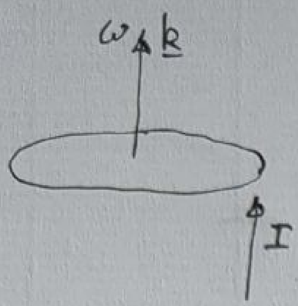


(d)

It's an AAA body so the body diagonal has moment of inertia $\frac{32ma^2}{6} = \frac{16ma^2}{3}$

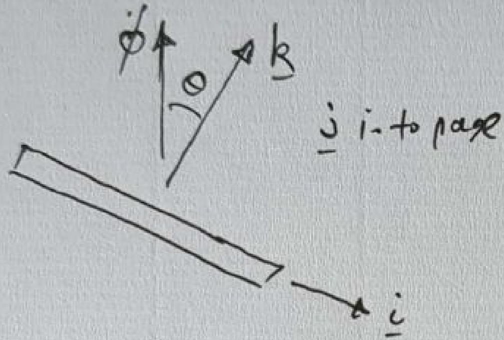
(This question was popular & well done)

2(a)



Impulse parallel to $\underline{k}$ so $\omega \underline{k}$ is unchanged.

Standard constant θ analysis using Gyro Equations



$$\omega_1 = \Omega_1 = -\dot{\phi}\sin\theta$$
$$\omega_2 = \Omega_2 = \dot{\theta}$$
$$\Omega_3 = \dot{\phi}\cos\theta$$
$$\omega_3 (\neq \Omega_3)$$

Steady wobble (nutation) $\therefore$ θ const $\therefore$ $\Omega_2 = 0$

Gyro equation 2 with $Q_2 = 0$

$$0 = A\cancel{\dot{\Omega}_2} + (A\Omega_3 - C\omega_3)\Omega_1$$

and we know Ω_1 is not zero (for non trivial solution)

$$\therefore \quad A\Omega_3 = C\omega_3$$

$$\therefore \quad \dot{\phi} = \frac{C\omega_3}{A\cos\theta}$$

Thin disc $\therefore$ $C = 2A$ (perpendicular axis theorem)

and for small θ and $\omega_3 = \omega$

$$\boxed{\dot{\phi} = 2\omega}$$ wobble rate

(b) We have above
$$\omega_1 = -\dot{\phi}\sin\theta$$
$$\omega_2 = \dot{\theta} = 0$$
$$\omega_3 = \omega$$
$$\Omega_3 = \dot{\phi}\cos\theta$$

For small θ this gives magnitudes of angular velocity

$$\boxed{\begin{aligned}\omega_1 &= \dot{\phi}\,\theta = 2\omega\theta \\ \omega_3 &= \omega\end{aligned}}$$

2(b) cont. The impulse I has a moment Ia about the $\underline{i}$ axis so the change in angular momentum about $\underline{i}$ is equal to Ia

$$\Delta \underline{h} = \underline{h}_{after} - \underline{h}_{before}$$

$$= A\omega_1 \underline{i} + \cancel{A\omega_2 \underline{j}} + C\omega \underline{k} - C\omega \underline{k}$$

$$= A\omega_1 \underline{i}$$

so $A\omega_1 = Ia$ and for a disc $A = \frac{1}{4}ma^2$

$$\therefore \frac{1}{4}ma^2\, 2\omega\theta = Ia$$

$$\therefore \boxed{\theta = \frac{2I}{ma\omega}}$$ $\left(\text{or } \frac{Ia}{2A\omega} \text{ or } \frac{Ia}{C\omega}\right)$ equally acceptable

(c) Translation $\underset{\sim}{I} = \Delta \underset{\sim}{p}$ Impulse = change of momentum

$$\therefore I = mv$$

$$\Delta(KE) = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{I}{m}\right)^2 = \boxed{\frac{I^2}{2m}}$$

wobble, $\Delta KE = KE_{final} - KE_{initial}$

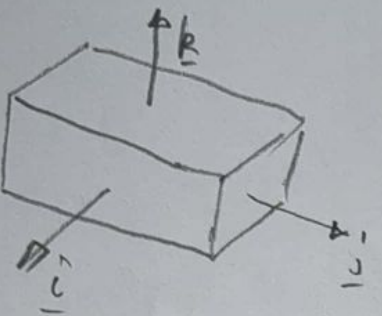
$$= \frac{1}{2}A\omega_1^2 + \cancel{\frac{1}{2}A\omega_2^2} + \frac{1}{2}C\omega_3^2 - \frac{1}{2}C\omega^2$$ $(\omega_3 = \omega)$

$$= \frac{1}{2}\frac{1}{4}ma^2\left(\frac{Ia}{\frac{1}{4}ma^2}\right)^2$$

$$\boxed{\Delta KE = \frac{2I^2}{m}}$$ $\left(\text{or } \frac{I^2a^2}{C} \text{ or } \frac{I^2a^2}{2A} \text{ or } \frac{ma^2\omega^2\theta^2}{2}\right)$

This question was not popular — maybe looked scary. Several thought it was a wobbling disc on a table. Can't be — no gravity! But those that got into it did well.

3(a)(i) $\underline{Q} = \dot{\underline{h}}$ ① $\underline{h} = A\omega_1 \underline{i} + B\omega_2 \underline{j} + C\omega_3 \underline{k}$



Angular velocity $\underline{\omega} = \omega_1 \underline{i} + \omega_2 \underline{j} + \omega_3 \underline{k}$

$$\dot{\underline{h}} = \dot{\underline{h}}\Big|_{rot} + \underline{\omega} \times \underline{h} \quad ②$$

$$\underline{\omega} \times \underline{h} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ A\omega_1 & B\omega_2 & C\omega_3 \end{vmatrix}$$

$$= (C\omega_2\omega_3 - B\omega_3\omega_2)\underline{i} - (C\omega_1\omega_3 - A\omega_3\omega_1)\underline{j} + (B\omega_1\omega_2 - A\omega_2\omega_1)\underline{k} \quad ③$$

With $\underline{Q} = Q_1 \underline{i} + Q_2 \underline{j} + Q_3 \underline{k}$ assemble ① ② & ③ to get

$$\boxed{\begin{aligned} Q_1 &= A\dot{\omega}_1 - (B-C)\omega_2\omega_3 \\ Q_2 &= B\dot{\omega}_2 - (C-A)\omega_3\omega_1 \\ Q_3 &= C\dot{\omega}_3 - (A-B)\omega_1\omega_2 \end{aligned}}$$

(ii) As above but the reference frame angular velocity $\underline{\Omega} = \Omega_1 \underline{i} + \Omega_2 \underline{j} + \Omega_3 \underline{k}$ and for the AAC body $\underline{h} = A\Omega_1 \underline{i} + A\Omega_2 \underline{j} + C\omega_3 \underline{k}$

with $\underline{\omega} = \Omega_1 \underline{i} + \Omega_2 \underline{j} + \omega_3 \underline{k}$

$$\dot{\underline{h}} = \dot{\underline{h}}_{rot} + \underline{\Omega} \times \underline{h}$$

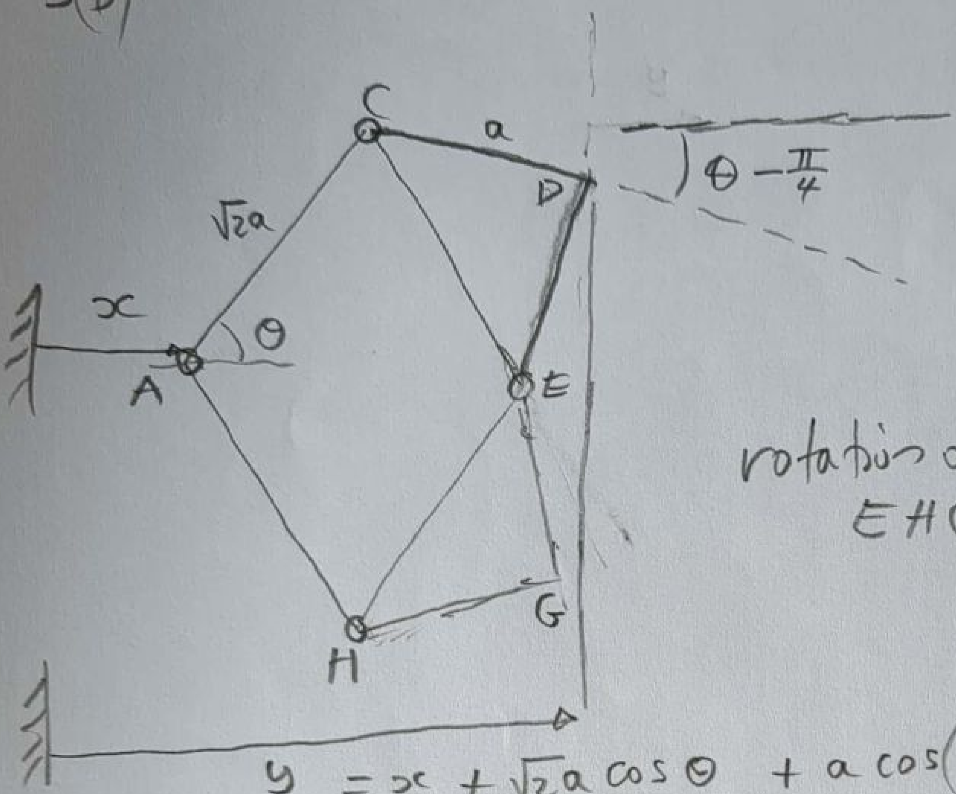
$$\underline{\Omega} \times \underline{h} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \Omega_1 & \Omega_2 & \Omega_3 \\ A\Omega_1 & A\Omega_2 & C\omega_3 \end{vmatrix}$$

$$= (C\omega_3 - A\Omega_3)\Omega_2 \underline{i} - (C\omega_3 - A\Omega_2)\underline{j}$$

$\therefore$ $Q_1 = A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2$
$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$
$Q_3 = C\dot{\omega}_3$

Bookwork, generally well done. But (ii) can't be obtained from (i) – you have to start over.

3(b)



rotation of triangle EHG is θ

$$y = x + \sqrt{2}a\cos\theta + a\cos\left(\theta - \frac{\pi}{4}\right)$$

$$dy = dx - \sqrt{2}a\sin\theta\, d\theta - a\sin\left(\theta - \frac{\pi}{4}\right)d\theta$$

$$= dx - a\left(\sqrt{2}\sin\theta + \sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4}\right)d\theta$$

$$= dx - a\sqrt{2}\left(\sin\theta + \frac{1}{2}\sin\theta - \frac{1}{2}\cos\theta\right)d\theta$$

$$= dx - \frac{a\sqrt{2}}{2}\left(3\sin\theta - \cos\theta\right)d\theta$$

$$dw = Q_x\, dx + Q_\theta\, d\theta$$

$$= F\, dy + M\, d\theta$$

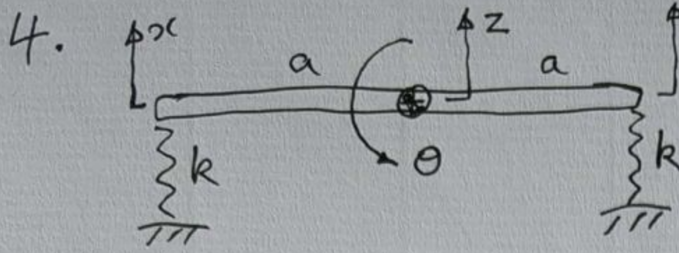
$$= F\left(dx - \frac{a\sqrt{2}}{2}(3\sin\theta - \cos\theta)\, d\theta\right) + M\, d\theta$$

$$\therefore \quad Q_x = F$$

$$Q_\theta = -\frac{Fa\sqrt{2}}{2}\left(3\sin\theta - \cos\theta\right) + M$$

Done well, but many just left this half of Q3 out and therefore couldn't get more than 50%. Tricky geometry, be careful.

4.



Note: $z = \frac{x+y}{2}$, $x = z - a\theta$

$\theta = \frac{y-x}{2a}$, $y = z + a\theta$

$I_G = \frac{1}{3} m a^2$

(a)(i) & (ii)

$$T = \frac{1}{2} m \dot{z}^2 + \frac{1}{2}\frac{1}{3} m a^2 \dot{\theta}^2$$

$$= \frac{1}{2} m \left(\frac{\dot{x}+\dot{y}}{2}\right)^2 + \frac{1}{2}\frac{1}{3} m a^2 \left(\frac{\dot{y}-\dot{x}}{2a}\right)^2$$

$$= \frac{1}{2} m \left[\frac{1}{4}\left(\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2\right) + \frac{1}{12}\left(\dot{x}^2 - 2\dot{x}\dot{y} + \dot{y}^2\right)\right]$$

$$= \frac{1}{2} m \left[\frac{1}{3}\dot{x}^2 + \frac{1}{3}\dot{y}^2 + 2\,\frac{1}{6}\dot{x}\dot{y}\right] \quad \text{(ans.)}$$

$$\therefore [M] = m \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad \text{(ans.)}$$

$$V = \frac{1}{2} k x^2 + \frac{1}{2} k y^2 \quad \text{(ans.)}$$

$$\therefore [K] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \quad \text{(ans.)}$$

(iii) $p_x = \frac{\partial T}{\partial \dot{x}} = \frac{1}{3} m \dot{x} + \frac{1}{6} m \dot{y}$ ans., $p_y = \frac{\partial T}{\partial \dot{y}} = \frac{1}{3} m \dot{y} + \frac{1}{6} m \dot{x}$ (ans.)

(iv) Need $[M]^{-1} = \left[\frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]^{-1} = \frac{6}{m}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \frac{2}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$T = \frac{1}{2}\begin{bmatrix} p_x & p_y \end{bmatrix} \frac{2}{m}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \end{bmatrix} \qquad V = \frac{1}{2} k x^2 + \frac{1}{2} k y^2$$

$$= \frac{1}{m}\left[2 p_x^2 - 2 p_x p_y + 2 p_y^2\right]$$

$$H = T + V = \frac{2}{m}\left(p_x^2 + p_y^2 - p_x p_y\right) + \frac{1}{2} k \left(x^2 + y^2\right) \quad \text{(ans.)}$$

(v) Hamilton: $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\therefore$ $\dot{x} = \frac{4 p_x}{m} - \frac{2 p_y}{m}$ (ans.)

$\dot{y} = \frac{4 p_y}{m} - \frac{2 p_x}{m}$ (ans.)

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $\therefore$ $\dot{p}_x = -kx$ (ans.)

$\dot{p}_y = -ky$ (ans.)

(b) (i) Use $G_2 = P_z \frac{x+y}{2} + P_\theta \frac{y-x}{2a}$ as given

2nd equation: $Q_z = \frac{\partial G_2}{\partial P_z} = \frac{x+y}{2} = z$ (ans.)

and $Q_\theta = \frac{\partial G_2}{\partial P_\theta} = \frac{y-x}{2a} = \theta$ (ans.)

4(b)(ii) 1st equation $p_x = \frac{\partial G_2}{\partial x} = \frac{1}{2} P_z - \frac{1}{2a} P_\theta$ (ans.)

and $p_y = \frac{\partial G_2}{\partial y} = \frac{1}{2} P_z + \frac{1}{2a} P_\theta$ (ans.)

(iii) $K = H + \frac{\partial G_2}{\partial t}$ but G_2 isn't explicitly a function of t

So $K = H$ and we have H given in 4(a)(iv)

$$\therefore K = \frac{2}{m}\left[\left(\tfrac{1}{2} P_z - \tfrac{1}{2a} P_\theta\right)^2 + \left(\tfrac{1}{2} P_z + \tfrac{1}{2a} P_\theta\right)^2 - \left(\tfrac{1}{2} P_z - \tfrac{1}{2a} P_\theta\right)\left(\tfrac{1}{2} P_z + \tfrac{1}{2a} P_\theta\right)\right] + \frac{k}{2}\left((z - a\theta)^2 + (z + a\theta)^2\right)$$

$$= \frac{1}{2m}\left[2P_z^2 + \frac{2}{a^2} P_\theta^2 - \left(P_z^2 - \frac{1}{a^2} P_\theta^2\right)\right] + \frac{k}{2}\left(2z^2 + 2a^2\theta^2\right)$$

$$= \frac{1}{2m}\left[P_z^2 + \frac{3}{a^2} P_\theta^2\right] + k\left(z^2 + (a\theta)^2\right) \quad \text{(ans)}$$

(iv) Hamilton : $\dot{z} = \frac{P_z}{m}$ (ans.) $\quad \dot{\theta} = \frac{3}{ma^2} P_\theta$ (ans.)

$\dot{P}_z = -2kz$ (ans.) $\quad \dot{P}_\theta = -2ka^2\theta$ (ans.)

This question was popular & pretty well done. Part IB Mechanics Lagrange was a good rehearsal. It's all about "cranking the handle" and getting on with it. The 3C5 Data Sheet is an invaluable guide.

Along the way there are lots of ways of checking your answers. eg check that 4(b)iii is $T+V$ and 4(b)(ii) use $P_z = m\dot{z}$ & $P_\theta = \frac{1}{3} ma^2\dot{\theta}$ to check against your answers for 4(a)(iii)