Version HEMH/5

EGT2
ENGINEERING TRIPOS PART IIA

Thursday 27 April 2023 2 to 3.40

Module 3C5

DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics and 3C6 Vibration datasheet 2023 (7 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A large cube of mass $8m$ and side $2a$ is assembled from two identical solid pieces, Piece 1 and Piece 2, images of which are shown in Fig. 1. Piece 1 and Piece 2 are each made up of four small cubes. The small cubes each have mass m and side a . An x, y, z coordinate system with origin O is shown in the figure. The centres of the cubes of Piece 1 are at $[a/2, a/2, -a/2]$,$[a/2, -a/2, -a/2]$, $[-a/2, -a/2, -a/2]$ and $[-a/2, -a/2, a/2]$. The centre of the assembled large cube is at O.

(a) The inertia matrix at O for Piece 1 is

$$k \begin{bmatrix} * & * & * \\ * & * & * \\ 3 & * & 16 \end{bmatrix}.$$

Find, in terms of m and a, the value of k and find the missing entries marked $*$. [50%]

(b) Find the inertia matrix at O for Piece 2. [20%]

(c) By adding the inertia matrices for Piece 1 and Piece 2, find the inertia matrix at O for the large cube, and verify your result. [20%]

(d) Find the moment of inertia of the large cube about its body diagonal. [10%]

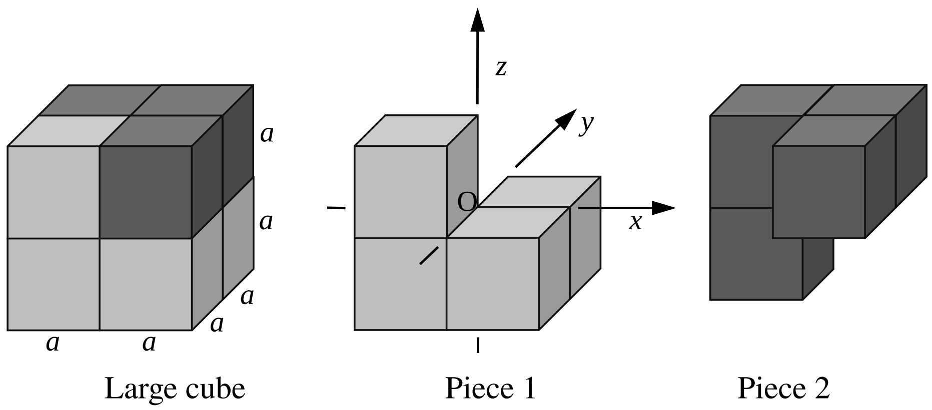

Fig. 1

2 A uniform thin circular disc of mass m and radius a is spinning about its symmetry axis $\mathbf{k}$ with angular velocity ω as shown in Fig. 2. The centre of the spinning disc is at rest, floating in free space. A small impulse I aligned parallel to $\mathbf{k}$ is delivered to the edge of the disc. The disc begins to wobble (free nutation) such that $\mathbf{k}$ traces out a cone with cone half angle θ, where θ is small.

(a) Use the Gyroscope Equations on the Data Sheet to determine the wobbling frequency of the free-spinning disc. [30%]

(b) Show that the angular velocity of the disc may be described by two orthogonal components whose magnitudes are $\omega_1 \approx 2\omega\theta$ and $\omega_3 = \omega$. Hence determine the wobble angle θ in terms of the applied impulse I. [40%]

(c) The kinetic energy of the disc has increased on account of the impulse. Show that the increase of kinetic energy associated with translation of the disc is

$$\frac{I^2}{2m}$$

and find the increase of kinetic energy associated with wobbling. [30%]

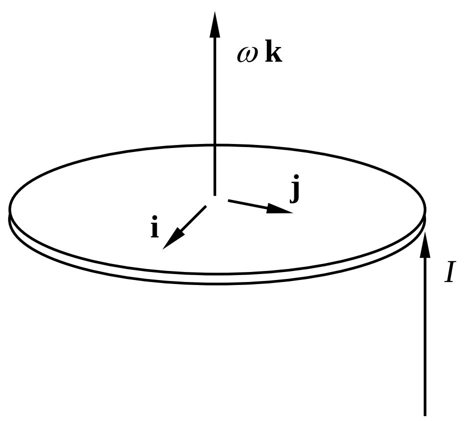

Fig. 2

3 (a) Derive from first principles (*i.e.* beginning from $Q = \dot{h}$) :

(i) Euler's equations for motion of a rigid body ; [25%]

(ii) the Gyroscope Equations for the motion of an axisymmetric "AAC" body [25%]

as given in Section 1.3 of the Datasheet.

(b) A planar mechanism comprises four identical rigid right-triangular plates ABC, CDE, EGH and HJA connected together with frictionless pivots at A, C, E and H as shown in Fig. 3. The side lengths of each of the plates are a, a and $\sqrt{2}a$. Points A and E are constrained to move horizontally. The mechanism has two degrees of freedom using generalized coordinates x and θ shown in the figure. A horizontal force F acts at D and a moment M acts at G. Find the generalized forces Q_x and Q_θ for the mechanism. [50%]

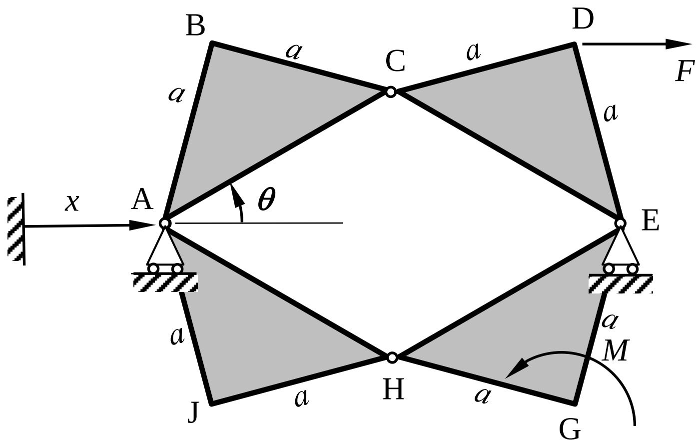

Fig. 3

4 A uniform beam AB of mass m and length $2a$ is supported at A and B by two springs of stiffness k as shown in Fig. 4. Motion is described by small displacements x and y of points A and B respectively. Gravity can be neglected. An alternative set of coordinates z and θ describe the translation and the rotation of the beam at its centre.

(a) Using the x, y coordinate system:

(i) find expressions for the kinetic energy and the potential energy;

(ii) find the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$;

(iii) find the generalized momenta p_x and p_y;

(iv) show that the Hamiltonian is $\frac{2}{m}(p_x^2 + p_y^2 - p_x p_y) + \frac{k}{2}(x^2 + y^2)$;

(v) Use Hamilton's equations to find four equations of motion in x and y. [50%]

Note from the Data Sheet the special case when the kinetic energy is expressible using a mass matrix $\boldsymbol{M}$:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1} \boldsymbol{p} \qquad \text{and} \qquad H = T + V$$

(b) In order to find the Kamiltonian using coordinates z and θ a canonical transformation of Type 2 is defined as

$$G_2(\boldsymbol{q}, \boldsymbol{P}, t) = P_z \frac{x+y}{2} + P_\theta \frac{y-x}{2a}$$

.

(i) Use the Type 2 second equation (as on the Data Sheet table of Canonical Transforms) to verify that $Q_z = z$ and $Q_\theta = \theta$;

(ii) Use the Type 2 first equation to find expressions for p_x and p_y in terms of P_z and P_θ.

(iii) Find the Kamiltonian $K(P_z, P_\theta, z, \theta)$;

(iv) Use Hamilton's equations to find four equations of motion in z and θ. [50%]

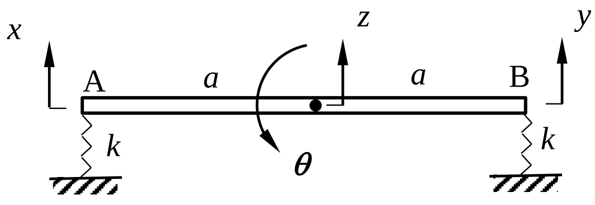

Fig. 4

5 Answers:

(a) Q1

$$(a)\quad \frac{ma^2}{6}\begin{bmatrix}16 & -3 & 3\\ -3 & 16 & 0\\ 3 & 0 & 16\end{bmatrix} \qquad (b)\quad \frac{ma^2}{6}\begin{bmatrix}16 & 3 & -3\\ 3 & 16 & 0\\ -3 & 0 & 16\end{bmatrix} \qquad (c)\quad \frac{16ma^2}{3} \qquad (d)\quad \frac{16ma^2}{3}.$$

(b) Q2

$$(a)\quad 2\omega \qquad (b)\quad \frac{2I}{ma\omega} \qquad (c)\quad \frac{2I^2}{m}.$$

(c) Q3

$$(b)\qquad Q_x = F \qquad Q_\theta = M + \frac{Fa\sqrt{2}}{2}(\cos\theta - 3\sin\theta).$$

(d) Q4

$$(a)\qquad T = \frac{m}{6}(\dot{x}^2 + \dot{y}^2 + \dot{x}\dot{y}) \qquad \boldsymbol{M} = \frac{m}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \qquad V = \frac{k}{2}(x^2 + y^2) \qquad \boldsymbol{K} = \begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$$

$$p_x = \frac{1}{3}m\dot{x} + \frac{1}{6}m\dot{y} \qquad p_y = \frac{1}{3}m\dot{y} + \frac{1}{6}m\dot{x}$$

$$\dot{x} = \frac{4p_x}{m} - \frac{2p_y}{m} \qquad \dot{y} = \frac{4p_y}{m} - \frac{2p_x}{m} \qquad \dot{p_x} = -kx \qquad \dot{p_y} = -ky$$

$$(b)\qquad p_x = \frac{1}{2}P_z - \frac{1}{2a}P_\theta \qquad p_y = \frac{1}{2}P_z + \frac{1}{2a}P_\theta \qquad K = \frac{1}{2m}(P_z^2 + \frac{3}{a^2}P_\theta^2) + k(z^2 + (a\theta)^2)$$

$$\dot{z} = \frac{P_z}{m} \qquad \dot{\theta} = \frac{3P_\theta}{ma^2} \qquad \dot{P}_z = -2kz \qquad \dot{P}_\theta = -2ka^2\theta$$

END OF PAPER