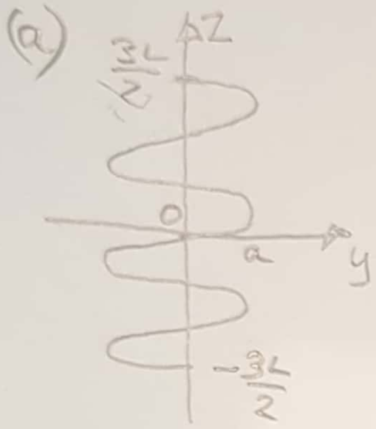
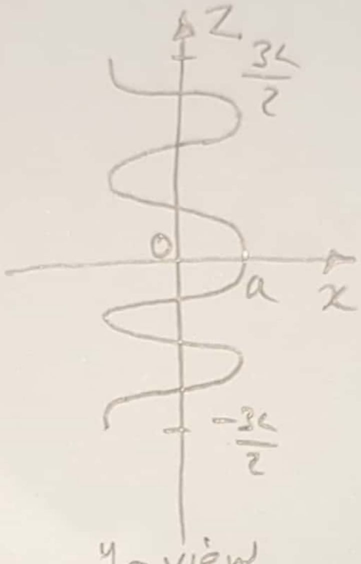
3C5 2014

Q1

(a)

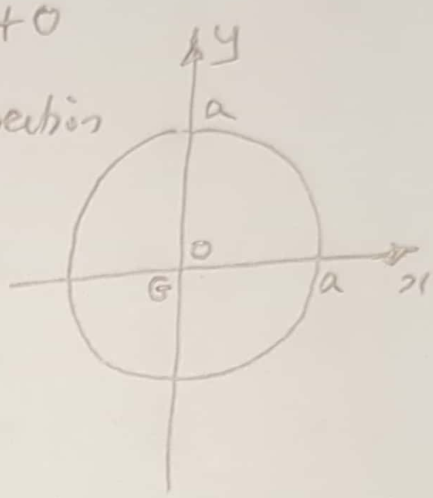


x-view



y-view

G at O by inspection



z-view

$$I_{xx} = \int (y^2+z^2)\,dm$$

$$= \int_{-3\pi}^{3\pi} \left(a^2\sin^2\theta + \frac{L^2\theta^2}{4\pi^2}\right)\frac{m}{2\pi}\,d\theta$$

$$= \left(a^2\frac{6\pi}{2} + \frac{L^2}{4\pi^2}\left[\frac{1}{3}\theta^3\right]_{-3\pi}^{3\pi}\right)\frac{m}{2\pi}$$

$$= \frac{3}{2}ma^2 + \frac{L^2}{12\pi^3}\left(27\pi^3+27\pi^3\right)\frac{m}{2\pi}$$

$$\boxed{I_{xx} = \frac{3}{2}ma^2 + \frac{9}{4}mL^2}$$

$$I_{yy} = \int (x^2+z^2)\,dm$$

= same as I_{xx} because $\int_{-3\pi}^{3\pi}\cos^2\theta\,d\theta = \int_{-3\pi}^{3\pi}\sin^2\theta\,d\theta$

$$\boxed{I_{yy} = \frac{3}{2}ma^2 + \frac{9}{4}mL^2}$$

$$I_{zz} = \int (x^2+y^2)\,dm$$

$$\boxed{I_{zz} = 3ma^2}$$

(looks like a hoop)

$$\boxed{I_{xy} = 0}$$

by inspection (for a hoop)

$$I_{yz} = \int yz\,dm$$

$$= \int_{-3\pi}^{3\pi} a\sin\theta\,\frac{L\theta}{2\pi}\,\frac{m}{2\pi}\,d\theta$$

$$= \frac{mLa}{4\pi^2}6\pi$$

$$\boxed{I_{yz} = \frac{3}{2\pi}mLa}$$

$$I_{xz} = \int xz\,dm$$

$$= \int_{-3\pi}^{3\pi} a\cos\theta\,\frac{L\theta}{2\pi}\,\frac{m}{2\pi}\,d\theta$$

$$\boxed{I_{xz} = 0}$$

$$I_G = \frac{3}{2}ma^2\begin{bmatrix} 1+\frac{3}{2}\left(\frac{L}{a}\right)^2 & 0 & 0 \\ 0 & 1+\frac{3}{2}\left(\frac{L}{a}\right)^2 & -\frac{L}{\pi a} \\ 0 & -\frac{L}{\pi a} & 2 \end{bmatrix}$$

(b) put $l = a$ $\quad \therefore I_G = \frac{3}{2}ma^2 \begin{bmatrix} 1+\frac{3}{2} & 0 & 0 \\ 0 & 1+\frac{3}{2} & -\frac{1}{\pi} \\ 0 & -\frac{1}{\pi} & 2 \end{bmatrix}$

$$= \frac{3}{4}ma^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -\frac{2}{\pi} \\ 0 & -\frac{2}{\pi} & 4 \end{bmatrix}$$

(c) $\underline{h} = [I_G]\underset{\sim}{\omega}$, $\dot{\underline{h}} = [I_G]\dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \underline{h}$

$\dot{\underset{\sim}{\omega}} = 0$ (const $\underline{\omega}$) so $\dot{\underline{h}} = \underline{\omega} \times \underline{h}$

$$\underline{h} = \frac{3}{4}ma^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -\frac{2}{\pi} \\ 0 & -\frac{2}{\pi} & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} = \frac{3}{4}ma^2 \begin{bmatrix} 0 \\ -\frac{2\Omega}{\pi} \\ 4\Omega \end{bmatrix}$$

$$\underline{Q} = \dot{\underline{h}} = \underline{\omega} \times \underline{h} = \Omega \underline{k} \times \left(-\frac{2\Omega}{\pi}\underline{j} + 4\Omega\underline{k} \right) \frac{3}{4}ma^2$$

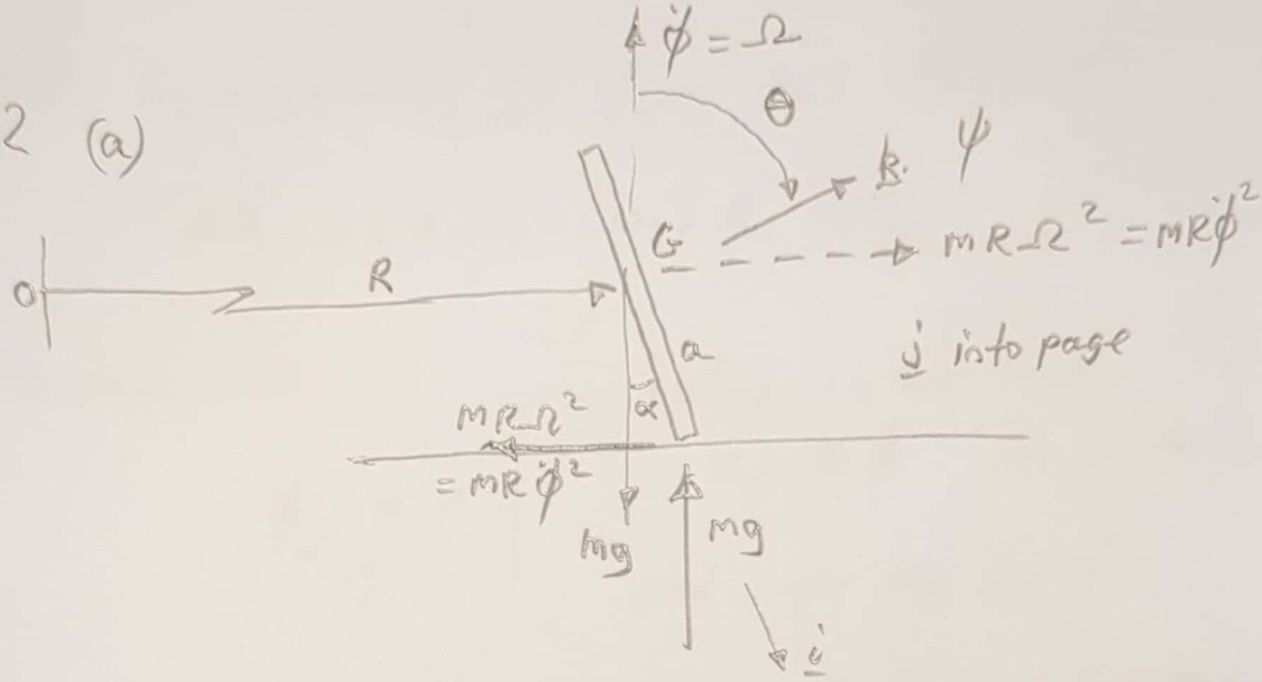
$$= \frac{2\Omega^2}{\pi} \frac{3}{4} ma^2 \underline{i}$$

$$= \frac{3}{2\pi} ma^2 \Omega^2 \underline{i}$$

Magnitude of couple is $\underline{\underline{\frac{3}{2\pi} ma^2 \Omega^2}}$

Check units correct: $kg\, m^2 s^{-2}$
$= N\, m$

Q2 (a)



Couple $Q_2 \, \underline{j} = \left(mR\dot{\phi}^2 a \cos\alpha - mga\sin\alpha \right) \underline{j}$

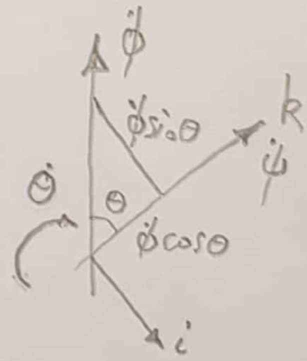
small α $\quad Q_2 = ma\left(R\Omega^2 - g\,\alpha \right)$

(b) Euler's angles θ, ϕ, ψ shown above

$$\omega_1 = -\dot{\phi}\sin\theta$$

$$\omega_2 = \dot{\theta}$$

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$



$\underline{j}$ into page

Reference frame

$$\Omega_1 = \omega_1$$

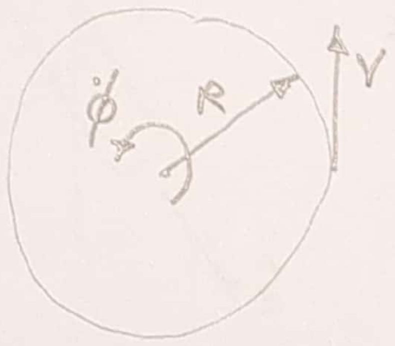
$$\Omega_2 = \omega_2$$

$$\Omega_3 = \dot{\phi}\cos\theta$$

(c) Gyro equation ② from 3CS datasheet

$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2$$

Top view

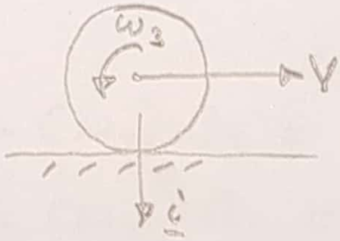


rolling on big circle

$$V = R\dot{\phi} \quad = \text{constant}$$

and no slip for disc

$$\omega_3 = -\frac{V}{a} \qquad \left(\begin{array}{c}\text{fast spin} \\ \dot{\psi} \gg \dot{\phi}\end{array}\right)$$



$$\theta = \frac{\pi}{2} - \alpha \quad = \text{constant}$$

$$\dot{\theta} = 0 \qquad \therefore \dot{\Omega}_2 = 0$$

$$\Omega_3 = \dot{\phi}\cos\theta = \frac{V}{R}\sin\alpha \simeq \frac{V}{R}\alpha$$

$$\omega_3 = -\frac{V}{a}, \qquad \Omega_1 = -\frac{V}{R}\sin\theta = -\frac{V}{R}\cos\alpha \simeq -\frac{V}{R}$$

$$\therefore \text{Gyro ② gives} \quad 0 + \left(\underbrace{\cancel{A\frac{V}{R}\alpha}}_{\text{small }\alpha} + C\frac{V}{a}\right)\left(-\frac{V}{R}\right) = ma\left(R\dot{\phi}^2 - g\alpha\right) \qquad \left(\dot{\phi} = \frac{V}{R}\right)$$

$$\therefore \quad mag\alpha = maR\frac{V^2}{R^2} + C\frac{V^2}{aR}$$

$$\therefore \quad = (ma^2 + C)\frac{V^2}{aR}$$

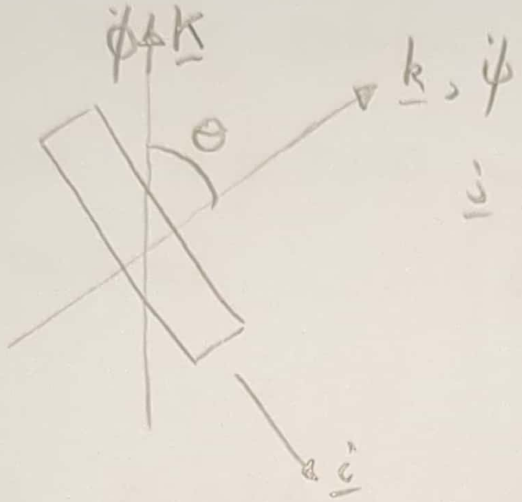
$$\therefore \boxed{R = \frac{(ma^2 + C)V^2}{ma^2 g\alpha}}$$

put $C = \frac{1}{2}ma^2$

$$\boxed{R = \frac{3}{2}\frac{V^2}{g\alpha}}$$

Q3 (a) Note that the gimbal is massless

(i) So this is a disc in free space



$\underline{j}$ into page

Euler angles θ, ϕ, ψ as shown

(ii) Gyro equations

No couple

$$A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 = Q_1 = 0 \quad (1)$$
$$A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 = Q_2 = 0 \quad (2)$$
$$C\dot{\omega}_3 = Q_3 = 0 \quad (3)$$

with
$$\Omega_1 = -\dot{\phi}\sin\theta$$
$$\Omega_2 = \dot{\theta}$$
$$\Omega_3 = \dot{\phi}\cos\theta$$
$$\omega_3 = \Omega_3 + \dot{\psi}$$

(3) $\rightarrow$ $\omega_3 = \text{const}$

The simplest way to get nutation frequency is to look at steady motion with constant small angle θ

Use gyro (2) with $\theta = \text{const}$, $\dot{\theta} = 0$
$\dot{\phi} = \text{const}$

$\therefore \quad 0 + \left(A\dot{\phi}\cos\theta - C\omega_3\right)\left(-\dot{\phi}\sin\theta\right) = 0$

Trivial solution $\theta = 0$ or π

else $A\dot{\phi}\cos\theta - C\omega_3 = 0$

$\cos\theta \simeq 1 \qquad \therefore \boxed{\dot{\phi} \simeq \frac{C\omega_3}{A}}$

iii/ $C = 2A$ thin disc $\dot{\phi} = 2\omega_3$

disc wobbles at twice spin rate

$C = A$ sphere $\dot{\phi} = \omega_3$

sphere wobbles at spin rate

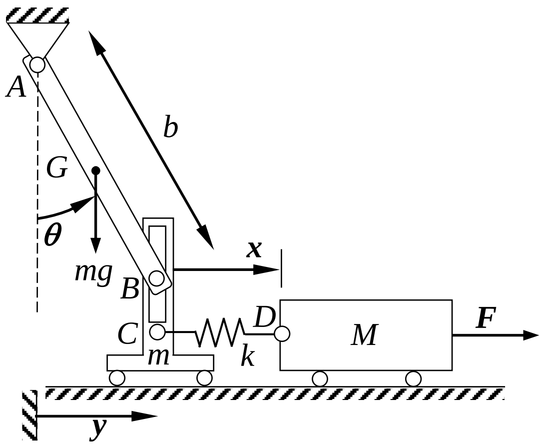
so spin & wobble are not distinguis

$\therefore$ sphere doesn't appear to wobble

3C5 - Questions 3b and 4 - AC685

Question 3b (AC685)

Given the system:



(i) Find an expression for the potential energy and determine the generalised forces Q_j associated with the generalised coordinates q_j when:

(a): $q_1 = x$, $q_2 = y$;

$$V = mg\frac{\sqrt{b^2 - y^2}}{2} \tag{1}$$

Consider a virtual change δq_j in the jth coordinate, consider the change in geometry arising δq_j, and calculate the work done by the applied forces:

$$\delta W = F(\delta x + \delta y) \tag{2}$$

Since

$$\delta W = Q_x \delta x + Q_y \delta y \tag{3}$$

It follows that

$$\begin{aligned} Q_x &= F \\ Q_y &= F \end{aligned} \tag{4}$$

(b) $q_1 = x$, $q_2 = \theta$.

$$V = mgb/2(1 - cos\,(\theta)) \tag{5}$$

$$\delta W = F(\delta x + bcos\,(\theta)\,\delta\theta) \tag{6}$$

Note: as in Lecture 12 the θ component can be computed starting from the velocity, or by first using the x and y coordinates, and then transforming into the correct coordinates. Since

$$\delta W = Q_x \delta x + Q_\theta \delta\theta \tag{7}$$

It follows that

$$\begin{aligned} Q_x &= F \\ Q_\theta &= Fbcos\left(\theta\right) \end{aligned} \tag{8}$$

(ii) How would the results in part (i) change if spring BC is replaced by a rigid link?

The system becomes a one DoF with $y = b \sin \theta$

For $q_1 = x$, $q_2 = y$, we can consider only $q = y$. It follows that

$$Q_y = F \tag{9}$$

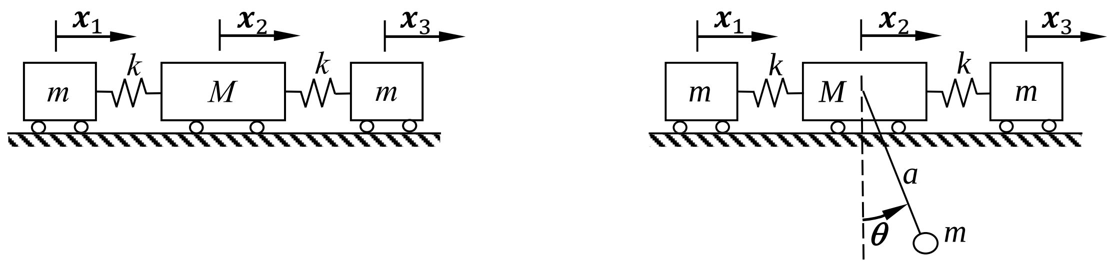
when

For $q_1 = x$, $q_2 = \theta$, we can consider only $q = \theta$

$$Q_\theta = Fbcos\left(\theta\right) \tag{10}$$

Question 4 (AC685)

Given the system:



(a) (b)

Fig. 4

(a) (i) [20%]

Evaluate the Hamiltonian of the system in Fig. 2(a).

The Kinetic Energy is:

$$T = \frac{1}{2}m\dot{x_1}^2 + \frac{1}{2}M\dot{x_2}^2 + \frac{1}{2}m\dot{x_3}^2 \tag{11}$$

The Potential Energy is:

$$V = \frac{1}{2}k\left(x_2 - x_1\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2 \tag{12}$$

$$H\left(\mathbf{p}, \mathbf{q}\right) = \sum_j p_j \dot{q}_j - T + V \tag{13}$$

$$p_1 = \frac{\partial T}{\partial \dot{x_1}} = m\dot{x}_1 \tag{14}$$

$$p_2 = \frac{\partial T}{\partial \dot{x_2}} = M\dot{x}_2 \tag{15}$$

$$p_3 = \frac{\partial T}{\partial \dot{x_3}} = m\dot{x}_3 \tag{16}$$

$$\begin{aligned} H\left(\mathbf{p}, \mathbf{q}\right) = p_1\dot{x_1} + p_2\dot{x_2} + p_3\dot{x_3} - \frac{1}{2}m\left(\frac{p_1}{m}\right)^2 - \frac{1}{2}M\left(\frac{p_2}{M}\right)^2 - \frac{1}{2}m\left(\frac{p_3}{m}\right)^2 \\ + \frac{1}{2}k\left(x_2 - x_1\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2 \end{aligned} \tag{17}$$

This can be rewritten as:

$$\begin{aligned} H\left(\mathbf{p}, \mathbf{q}\right) = -\frac{1}{2m}\left(p_1\right)^2 - -\frac{1}{2M}\left(p_2\right)^2 - \frac{1}{2m}\left(p_3\right)^2 \\ + \frac{1}{2}k\left(x_2 - x_1\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2 \end{aligned} \tag{18}$$

(a) (ii)[30%]
Use the Poisson brackets to assess if the total momentum is conserved.

$$p_{tot} = p_1 + p_2 + p_3 \tag{19}$$

$$\begin{aligned} \{p_{tot}, H\} = &\frac{\partial p_{tot}}{\partial x_1}\frac{\partial H}{\partial p_1} - \frac{\partial p_{tot}}{\partial p_1}\frac{\partial H}{\partial x_1} \\ &+\frac{\partial p_{tot}}{\partial x_2}\frac{\partial H}{\partial p_2} - \frac{\partial p_{tot}}{\partial p_2}\frac{\partial H}{\partial x_2} \\ &+\frac{\partial p_{tot}}{\partial x_3}\frac{\partial H}{\partial p_3} - \frac{\partial p_{tot}}{\partial p_3}\frac{\partial H}{\partial x_3} \end{aligned} \tag{20}$$

since $\frac{\partial p_{tot}}{\partial x_j} = 0, \forall j$ and $\frac{\partial p_{tot}}{\partial p_j} = 1, \forall j$

$$\{p_{tot}, H\} = -k\left(x_2 - x_1\right) + k\left(x_2 - x_1\right) - k\left(x_3 - x_2\right) + k\left(x_3 - x_2\right) = 0 \tag{21}$$

The total moment is conserved!

Justify the result obtained in point (ii): there are no external generalised force (i.e. under free motion, the generalised momentum is conserved).

(b) (i)[30%]
Derive the 4 equations of motion for the new system in Fig. 2(b) by using the Lagrange equation.

The additional potential kinetic energies are given by:

$$V_{bob} = mga\left(1 - \cos(\theta)\right) \tag{22}$$

$$T_{bob} = \frac{1}{2}mv_{bob}^2 \tag{23}$$

where

$$\begin{aligned} v_{bob} = &\left(-a\dot{\theta}\sin\left(\theta\right)\right)^2 + \left(\dot{x_2} + a\dot{\theta}\cos\left(\theta\right)\right)^2 \\ &= \dot{x_2}^2 + a^2\dot{\theta}^2 + 2a\dot{x_2}\dot{\theta}\cos\left(\theta\right) \end{aligned} \tag{24}$$

We will derive the 3 equations in x_j and one in θ using the updated kinetic and potential energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_j} - \frac{\partial V}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = 0 \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial V}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0 \tag{26}$$

Taking the derivatives

$$\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m\dot{x}_1 \tag{27}$$

$$\left(\frac{\partial T}{\partial \dot{x}_2}\right) = M\dot{x}_2 + m\dot{x}_2 + ma\dot{\theta}\cos(\theta) \tag{28}$$

$$\left(\frac{\partial T}{\partial \dot{x}_3}\right) = m\dot{x}_3 \tag{29}$$

$$\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ma\dot{x}_2\cos(\theta) + ma^2\dot{\theta} \tag{30}$$

$$\left(\frac{\partial V}{\partial \dot{x}_j}\right) = \left(\frac{\partial V}{\partial \dot{\theta}}\right) = 0 \tag{31}$$

$$\frac{\partial T}{\partial x_j} = 0 \tag{32}$$

$$\frac{\partial V}{\partial x_1} = -k(x_2 - x_1) \tag{33}$$

$$\frac{\partial V}{\partial x_2} = +k(x_2 - x_1) - k(x_3 - x_2) \tag{34}$$

$$\frac{\partial V}{\partial x_3} = k(x_3 - x_2) \tag{35}$$

$$\frac{\partial L}{\partial \theta} = -ma\dot{x}_2\dot{\theta}\sin(\theta) - mga\sin(\theta) \tag{36}$$

Leading to:

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) &= 0 \\ (M+m)\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) + ma\left(\ddot{\theta}\cos(\theta) - \dot{\theta}^2\sin(\theta)\right) &= 0 \\ m\ddot{x}_3 + k(x_3 - x_2) &= 0 \\ ma^2\ddot{\theta} + ma\ddot{x}_2\cos(\theta) + mga\sin(\theta) &= 0 \end{aligned} \tag{37}$$

(b) (ii) [20%]

Linearise the equation of motions considering small amplitude vibrations. Write the resulting set of equations in matrix form.

Consider that $\sin(\theta) \simeq \theta$, $\cos(\theta) \simeq 1$, and remove the nonlinear terms.$\sin(\theta) \simeq \theta$

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) &= 0 \\ (M+m)\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) + ma\ddot{\theta} &= 0 \\ m\ddot{x}_3 + k(x_3 - x_2) &= 0 \\ ma^2\ddot{\theta} + ma\ddot{x}_2 + mga\theta &= 0 \end{aligned} \tag{38}$$

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & M+m & 0 & ma \\ 0 & 0 & m & 0 \\ 0 & ma & 0 & ma^2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & k & 0 \\ 0 & 0 & 0 & mga \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$