Version HEMH/4

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 24 April 2024 9.30 to 11.10

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics and 3C6 Vibration datasheet 2023 (7 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A thin wire of mass $3m$ is wound to form a three-coil helical spring of radius a and length $3L$ as shown in Fig. 1. The shape of the coil is described in x, y, z coordinates by the parametric equations

$$x = a\cos\theta, \quad y = a\sin\theta \quad \text{and} \quad z = \frac{L\theta}{2\pi} \qquad \text{for} -3\pi \le \theta \le 3\pi.$$

(a) (i) Find the inertia matrix of the coil at its centre of mass G.

[Note that the mass distribution along the wire is $\mathrm{d}m = (m/2\pi)\mathrm{d}\theta$.
Also note the two definite integrals $\int_{-3\pi}^{3\pi} \theta\sin\theta\mathrm{d}\theta = 6\pi$ and $\int_{-3\pi}^{3\pi} \theta\cos\theta\mathrm{d}\theta = 0$]

(ii) For the case $L = a$ show that the inertia matrix of the coil at G is

$$\frac{3ma^2}{4}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & -\frac{2}{\pi} \\ 0 & -\frac{2}{\pi} & 4 \end{bmatrix}.$$

[60%]

(b) The spring in (a)(ii) is caused to spin steadily about the z axis at an angular velocity of $\Omega\mathbf{k}$. Use the result $\mathbf{Q} = \dot{\mathbf{h}}$ to find the magnitude of the couple that is required to sustain this steady motion. [40%]

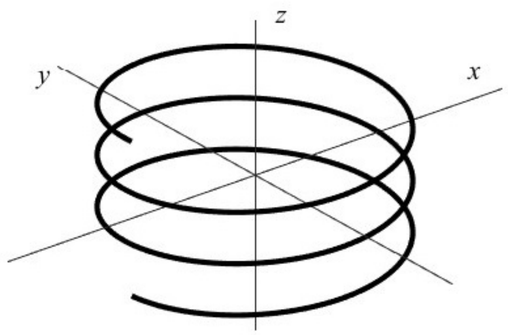


Fig. 1

2 A thin solid circular disc of radius a, mass m and moments of inertia A, A, C is rolling without slip on a horizontal floor. The forward speed of the disc is V and the point of contact P of the disc with the floor traces out a large circle of radius $R \gg a$ as shown in Fig. 2. The disc is inclined to the vertical by a small angle α.

(a) On a free-body diagram of the disc show all the forces acting on the disc and hence find an expression for the couple acting on the disc. [20%]

(b) Identify Euler's angles for motion of the disc and use these, with a no-slip condition, to write down expressions for the three components of angular velocity of the disc. [30%]

(c) Use the gyroscope equations or otherwise to find an expression for the path radius R in terms of the forward speed V and the tilt angle α. For the case $C = 2A = \frac{1}{2}ma^2$ show that $R = \frac{3V^2}{2g\alpha}$. [50%]

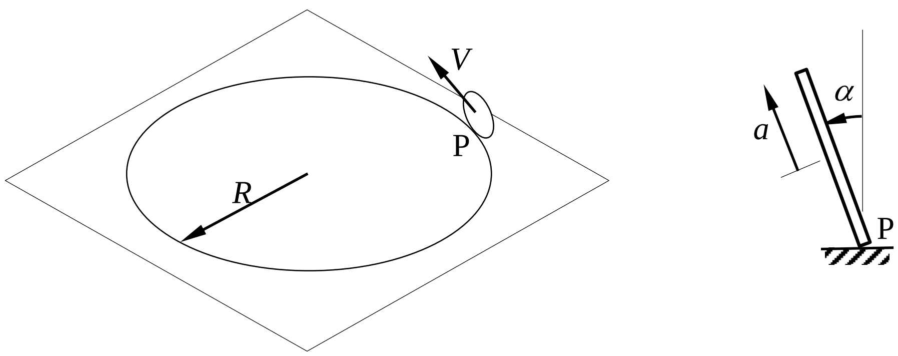


Fig. 2

3 (a) An "AAC" rotor is mounted in a massless gimbal as shown in Fig. 3(a). The rotor is spinning about its axis of symmetry with angular velocity ω and the spin axis is inclined at angle θ to a reference vertical as shown.

(i) Define Euler's angles for the rotor. [10%]

(ii) Using the Gyroscope Equations or otherwise find an expression for the frequency of small vibration if the rotor is subject to a small disturbance. [30%]

(iii) For your answer in (ii) comment on the special cases $C = 2A$ and $C = A$. [10%]

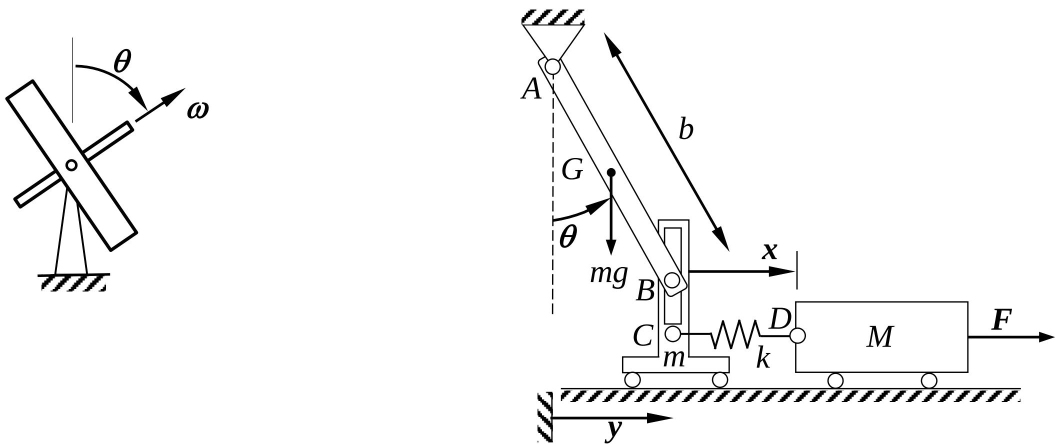
(b) A uniform rigid bar AB of mass m and length b is connected at B to a frictionless slider system of mass m, as shown in Fig. 3(b). The slider is in turn connected to a mass M via a spring CD of stiffness k. An external force F acts as shown. Angle θ describes the inclination of the bar from the vertical and y is the displacement of of point C. The extension of the spring is x.

(i) Find an expression for the potential energy and determine the generalised forces Q_j associated with the generalised coordinates q_j when:

$q_1 = x, q_2 = y$; [15%]

$q_1 = x, q_2 = \theta$. [25%]

(ii) How would the results in part (i) change if spring CD is replaced by a rigid link? [10%]



(a) (b)

Fig. 3

4 (a) Three masses m, M and m are connected to each other by springs of stiffness k, as shown in Fig. 4(a). Displacements x_1, x_2 and x_3 describe the motion of the three masses as shown.

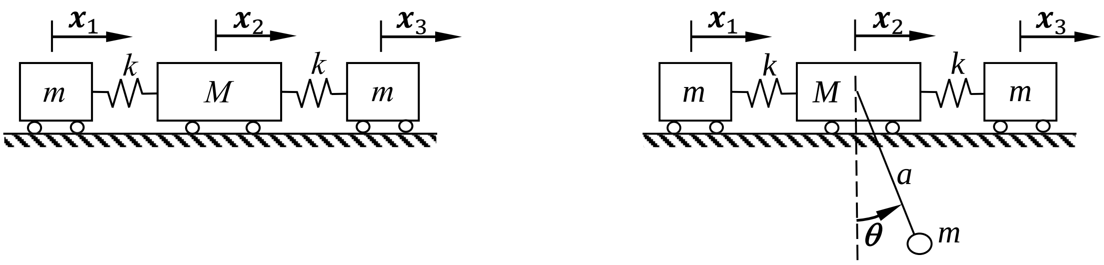
(i) Evaluate the Hamiltonian of the system. [20%]

(ii) Use Poisson brackets to determine whether the total momentum of the system is conserved. Justify your answer on physical grounds. [30%]

(b) An additional mass m is mounted with a pin joint to the central mass M through an inextensible link of length a as shown in Fig. 4(b). The generalised coordinate θ is introduced.

(i) Derive the four equations of motion for this new system by using the Lagrange equation. [30%]

(ii) Linearise the equations of motion considering small amplitude vibrations. Write the resulting set of equations in matrix form. [20%]



(a) (b)

Fig. 4

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ANSWERS

1 (a) $\frac{3ma^2}{2}\begin{bmatrix} 1+\frac{3L^2}{2a^2} & 0 & 0 \\ 0 & 1+\frac{3L^2}{2a^2} & -\frac{L}{\pi a} \\ 0 & -\frac{L}{\pi a} & 2 \end{bmatrix}.$

(b) Couple $=\frac{3}{2\pi}ma^2\Omega^2$

2 (a) $Q_2 = ma(R\Omega^2 - g\alpha)$ for small α

(b) $\Omega_1 = \omega_1 = -\dot{\phi}, \quad \Omega_2 = \omega_2 = -\dot{\alpha} = 0, \quad \Omega_3 = \dot{\phi}\alpha, \quad \omega_3 = -\frac{V}{a}$ for small α

(c) $R = \frac{(ma^2+C)V^2}{ma^2 g\alpha}$

3 (a) (i) $\Omega_1 = -\dot{\phi}\sin\theta, \quad \Omega_2 = \dot{\theta}, \quad \Omega_3 = \dot{\phi}\cos\theta$

(ii) $\dot{\phi} = \frac{C\omega_3}{A}$

(b) (i) $V = mg\frac{\sqrt{b^2-y^2}}{2}, \quad Q_x = F, \quad Q_y = F$

$V = \frac{mgb(1-\cos\theta)}{2}, \quad Q_x = F, \quad Q_\theta = Fb\cos\theta$

(ii) now only 1-dof, use θ: $V = \frac{mgb(1-\cos\theta)}{2}, \quad Q_\theta = Fb\cos\theta$

4 (a) $H(\mathbf{p},\mathbf{q}) = -\frac{1}{2m}(p_1)^2 - -\frac{1}{2M}(p_2)^2 - \frac{1}{2m}(p_3)^2 + \frac{1}{2}k(x_2-x_1)^2 + \frac{1}{2}k(x_3-x_2)^2$

(b) (i) $m\ddot{x}_1 + k(x_1 - x_2) = 0$

$(M+m)\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) + ma\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = 0$

$m\ddot{x}_3 + k(x_3 - x_2) = 0$

$ma^2\ddot{\theta} + ma\ddot{x}_2\cos\theta + mga\sin\theta = 0$

(ii) $\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & M+m & 0 & ma \\ 0 & 0 & m & 0 \\ 0 & ma & 0 & ma^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & k & 0 \\ 0 & 0 & 0 & mga \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$