Part IIA, 3C5 2025

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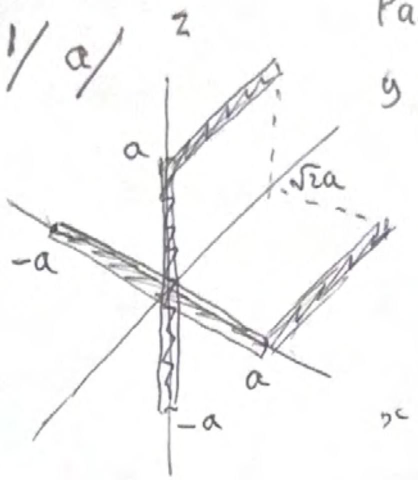
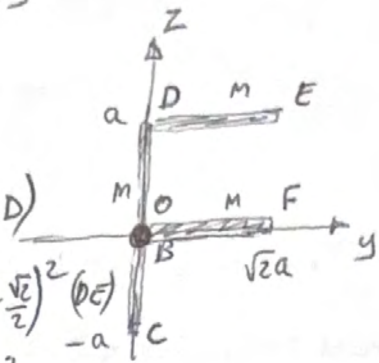


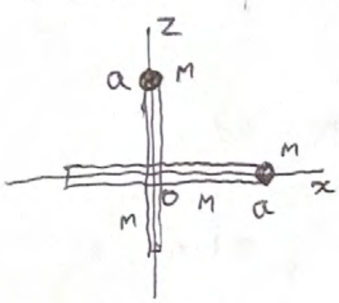
<u>x-view</u>



$$I_{xx} = \frac{1}{12}m(2a)^2 \quad (CD)$$
$$+\frac{1}{12}m(a\sqrt{2})^2 + ma^2\left(1+\frac{\sqrt{2}}{2}\right)^2 \quad (DE)$$
$$+\frac{1}{12}m(a\sqrt{2})^2 + ma^2\left(\frac{\sqrt{2}}{2}\right)^2 \quad (BF)$$
$$= ma^2\left(\frac{1}{3}+\frac{1}{6}+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}\right)$$
$$= \frac{8}{3}ma^2$$

$$I_{yz} = m\,a\,\frac{a\sqrt{2}}{2} = ma^2\frac{\sqrt{2}}{2}$$

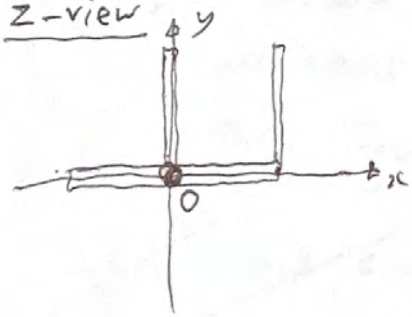
<u>y-view</u>



$$I_{yy} = \frac{1}{12}m(2a)^2 + \frac{1}{12}m(2a)^2 + ma^2 + ma^2$$
$$= \frac{8}{3}ma^2$$

$$I_{xz} = 0$$

<u>z-view</u>



Same as x-view

$$I_{zz} = \frac{8}{3}ma^2$$
$$I_{xy} = ma^2\frac{\sqrt{2}}{2}$$

so $I_{xx} = I_{yy} = I_{zz} = \frac{8}{3}ma^2$

N.B. this is NOT an AAA body because I_{yz} and I_{xy} are not zero. Need to use the whole inertia matrix for part (b)

$$I_0 = ma^2\begin{bmatrix} \frac{8}{3} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{8}{3} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{8}{3} \end{bmatrix}$$

(b) We're told that G is at $\frac{1}{4}(a, \sqrt{2}a, a)$

and that OG is a principal axis

Verify using

$[I_0]\underline{u} = \lambda \underline{u}$ where $\underline{u}$ is the principal axis
and λ is the principal moment of inertia

$$ma^2\begin{bmatrix} \frac{8}{3} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{8}{3} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{8}{3} \end{bmatrix}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = ma^2\begin{bmatrix} \frac{8}{3} - 1 \\ -\frac{\sqrt{2}}{2} + \frac{8\sqrt{2}}{3} - \frac{\sqrt{2}}{2} \\ -1 + \frac{8}{3} \end{bmatrix} = ma^2\begin{bmatrix} \frac{5}{3} \\ \sqrt{2}\left(\frac{8}{3} - 1\right) \\ \frac{5}{3} \end{bmatrix} = \frac{5}{3}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} ma^2$$

So yes, OG is principal and I_{OG} is $\frac{5}{3}ma^2$

note, don't assume its an "AAA" body just because $I_{xx} = I_{yy} = I_{zz} = \frac{8}{3}ma^2$ are all equal. If ~~they~~ it was an AAA body then I_{OG} would be $\frac{8}{3}ma^2$

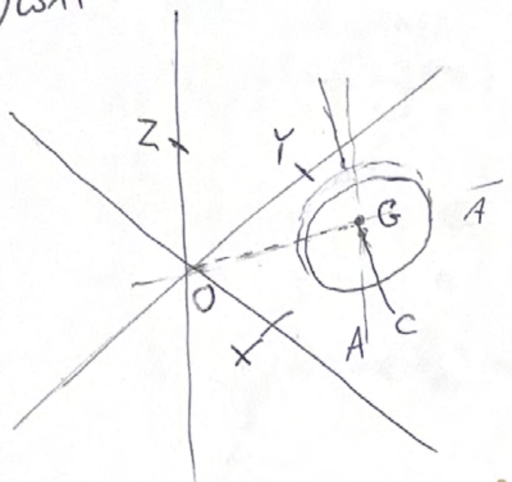
(c) ~~We're told that the framework is "AAC" at G~~ and ~~but we can deduce that the value "$\frac{5}{3}ma^2$" that~~ we found in (b) ~~cannot be~~ is not ~~the "C" value because if it were then by parallel axes theorem the body would be AAC~~

~~the~~ we have principal moments of inertia at G

as $\frac{5}{3}ma^2$, $\frac{5}{3}ma^2$, $\frac{8}{3}ma^2$ ~~so the~~ which is "AAC"

So the "C" value is $\frac{8}{3}ma^2$

(C) cont



So to find the principle moments of inertia at O use parallel axes theorm on A and on C

with $\overline{OG} = \sqrt{x^2+y^2+z^2}$

$$= \sqrt{\left(\frac{a}{4}\right)^2 + \left(\frac{a\sqrt{2}}{4}\right)^2 + \left(\frac{a}{4}\right)^2}$$

$$= \frac{a}{4}\left(1+2+1\right)^{\frac{1}{2}}$$

$$= \frac{a}{2}$$

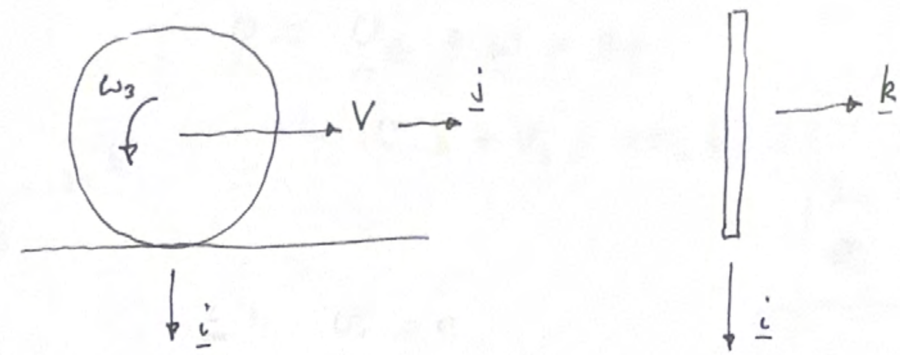
so at O principal moments of inertia

are $\underline{\underline{\frac{5}{3}Ma^2}}$, $\frac{5}{3}Ma^2 + 4M\left(\frac{a}{2}\right)^2$, $\frac{8}{3}Ma^2 + 4M\left(\frac{a}{2}\right)$

already found

$= \underline{\underline{\frac{8}{3}Ma^2}}$ $\quad = \underline{\underline{\frac{11}{3}Ma^2}}$

2 (a)



Steady state

$\underline{v} = V\,\underline{j}$ $\qquad v_1 = 0 \quad v_2 = V \quad v_3 = 0$

$\underline{\omega} = \omega_3\,\underline{k} = -\frac{V}{a}\,\underline{k}$ $\qquad \omega_1 = 0 \quad \omega_2 = 0 \quad \omega_3 = \frac{-V}{a}$

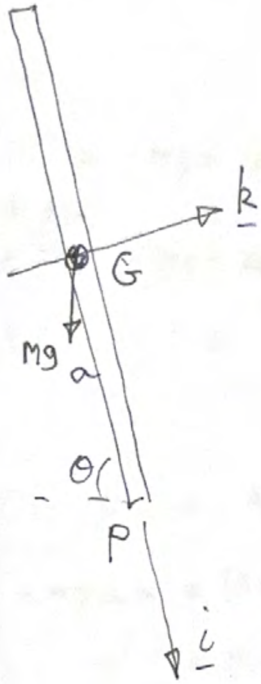
$\underline{F} = -mg\,\underline{i}$ $\qquad F_1 = -mg \quad F_2 = 0 \quad F_3 = 0$

$\underline{\Omega} = 0$ $\qquad \Omega_3 = 0$

This means that in perturbed motion $\left(\theta = \frac{\pi}{2} - \alpha\right)$ these quantities are small :

$\alpha, v_1, v_3, \omega_1, \omega_2, F_2, F_3, \Omega_3$

2 (b)(i) No slip at P



$$0 = \underline{v}_G + \underline{\omega} \times a\underline{i}$$

$$= (v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}) + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ a & 0 & 0 \end{vmatrix}$$

$\underline{i}$: $v_1 = 0$

$\underline{j}$: $v_2 + a\omega_3 = 0$

$\underline{k}$: $v_3 - a\omega_2 = 0$

$$\boxed{\begin{aligned} v_1 &= 0 \\ \therefore v_2 &= -a\omega_3 \\ \therefore v_3 &= a\omega_2 \end{aligned}}$$

(ii) $\underline{Q}_P = \dot{\underline{h}}_P + \dot{\underline{r}}_P \times \underline{p}$

$$\underline{p} = ma(-\omega_3 \underline{j} + \omega_2 \underline{k})$$

$$\dot{\underline{r}}_P = \underline{v}_G + \underline{\Omega} \times a\underline{i}$$

$$= (v_2 + a\Omega_3)\underline{j} + (v_3 - a\omega_2)\underline{k}$$

$$\dot{\underline{r}}_P \times \underline{p} = \left(ma\omega_2(v_2 + a\Omega_3) + ma\omega_3(v_3 - a\omega_2)\right)\underline{i}$$

$$= ma^2 \underline{i}\left(-\omega_2\omega_3 + \omega_2\Omega_3 + \cancel{\omega_3\omega_2} - \cancel{\omega_3\omega_2}\right)$$

$$= ma^2 \underline{i}\,(\Omega_3 - \omega_3)\,\omega_2$$

Fast spin $\omega_3 \gg \Omega_3$ $\therefore$ $\dot{\underline{r}}_P \times \underline{p} = -ma^2\omega_3\omega_2 \underline{i}$

$$\underline{h}_P = \underline{h}_G + (\underline{r}_G - \underline{r}_P) \times \underline{p}$$

$$= A\omega_1 \underline{i} + A\omega_2 \underline{j} + C\omega_3 \underline{k} \quad - (a\underline{i}) \times ma(-\omega_3 \underline{j} + \omega_2 \underline{k})$$

$$= A\omega_1 \underline{i} + (A\omega_2 + ma^2\omega_2)\underline{j} + (C\omega_3 + ma^2\omega_3)\underline{k}$$

$$= A\omega_1 \underline{i} + (A + ma^2)\omega_2 \underline{j} + (C + ma^2)\omega_3 \underline{k}$$

$$\dot{\underline{h}}_P = \dot{\underline{h}}_P\big|_{rot} + \underline{\Omega} \times \underline{h}_P$$

$\cancel{h_P}$ $\underline{\Omega} \times \underline{h}_P = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \Omega_3 \\ A\omega_1 & (A+ma^2)\omega_2 & (C+ma^2)\omega_3 \end{vmatrix}$

$\underline{Q}_P = -mga\cos\theta\ \underline{j}$

$$\therefore \begin{cases} ① \quad \underline{i}: & 0 = A\dot{\omega}_1 + (C+ma^2)\omega_2\omega_3 - (A+ma^2)\omega_2\Omega_3 - ma^2\omega_2\omega_3 \\ ② \quad \underline{j}: & -mga\cos\theta = (A+ma^2)\dot{\omega}_2 + A\omega_1\Omega_3 - (C+ma^2)\omega_1\omega_3 \\ ③ \quad \underline{k}: & 0 = (C+ma^2)\dot{\omega}_3 + (A+ma^2)\omega_1\omega_2 - A\omega_1\omega_2 \end{cases}$$

$$\therefore \begin{cases} ① & 0 = A\dot{\omega}_1 + C\omega_2\omega_3 \\ ② & -mga\alpha = (A+ma^2)\dot{\omega}_2 - (C+ma^2)\omega_1\omega_3 \\ ③ & 0 = (C+ma^2)\dot{\omega}_3 \end{cases}$$

$(\cancel{\omega_2}\ \Omega_3 \ll \omega_3$ ~~is small~~ $)$

$(\cos\theta = \sin\alpha = \alpha$
$\Omega_3 \ll \omega_3$

③ $\therefore \omega_3$ is constant

① $\therefore \dot{\omega}_1 = -\frac{C\omega_3}{A}\omega_2$

② $\therefore (A+ma^2)\ddot{\omega}_2 - (C+ma^2)\omega_3\dot{\omega}_1 + mga\dot{\alpha} = 0$

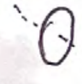
and $\omega_2 = \dot{\theta} = -\dot{\alpha}$

$\therefore (A+ma^2)\ddot{\omega}_2 + (C+ma^2)\omega_3\frac{C\omega_3}{A}\omega_2 - mga\,\omega_2 = 0$

$\therefore (A+ma^2)\ddot{\omega}_2 + \left[(C+ma^2)\frac{C}{A}\omega_3^2 - mga\right]\omega_2 = 0$

SHM if $\omega_3^2 > \dfrac{mga}{(C+ma^2)\frac{C}{A}}$

$C = ma^2$
$A = \frac{1}{2}ma^2$ $\quad \therefore \omega_3^2 > \dfrac{g}{4a}, \quad V^2 > \dfrac{ga}{4}$  bike wheel

$C = \frac{1}{2}ma^2$
$A = \frac{1}{4}ma^2$ $\quad \therefore \omega_3^2 > \dfrac{g}{3a}, \quad V^2 > \dfrac{ga}{3}$  disc

wobbling frequency for a wheel, $C = Ma^2$

$A = \frac{1}{2} Ma^2$

$$\therefore \left(\tfrac{1}{2}+1\right)\ddot{\omega}_2 + \left[(1+1)\,2\,\omega_3^2 - \frac{g}{a}\right]\omega_2 = 0$$

$$\therefore \boxed{\frac{3}{2}\ddot{\omega}_2 + \left[4\omega_3^2 - \frac{g}{a}\right]\omega_2 = 0}$$

$v_3 = a\,\omega_2 \qquad v_2 = -a\,\omega_3$

$$\therefore \boxed{\ddot{v}_3 + \frac{2}{3}\left[4V^2 - ga\right]v_3 = 0}$$

$$(\text{wobble frequency})^2 = \frac{2}{3}\left[4V^2 - ga\right]$$

$\therefore$ need $V^2 > \frac{ga}{4}$

3 (a) (i) The question gives the necessary angular velocities but need to add $\dot{\theta}$ to the $\underline{j}$ component

$$\Omega_1 = -\Omega \cos\lambda \sin\theta$$
$$\Omega_2 = \Omega \sin\lambda + \dot{\theta}$$
$$\Omega_3 = \Omega \cos\lambda \cos\theta$$

Gyro equations:

$$Q_1 = A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2$$
$$Q_2 = A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1$$
$$Q_3 = C\dot{\omega}_3$$

Fast spin and frictionless $\therefore \omega_3 = \omega = \text{const}$

$\therefore A\Omega_3 \ll C\omega_3$

$Q_2 = 0$ (free to turn about $\underline{j}$)

$$\therefore \quad A\dot{\Omega}_2 - C\omega_3\Omega_1 = 0$$

$$\therefore \quad \underline{A\ddot{\theta} + C\omega_3 \Omega \cos\lambda \sin\theta = 0}$$

(ii) Steady state, $\ddot{\theta} = 0 \quad \therefore \sin\theta = 0$

$\therefore \theta = 0$ or π

$\theta =$ small (around zero) $\quad \therefore \quad A\ddot{\theta} + C\omega_3\Omega\cos\lambda\,\theta = 0$

This is SHM and therefore stable when $\theta = 0$, aligned with North

3(a)(ii) cont

Now try $\theta = \pi + x$ (around $\theta = \pi$, x small)

$$\sin\theta = \sin(\pi + x) = -\sin x$$

$$\therefore \quad A\ddot{x} - C\omega_3 \Omega \cos\lambda \, x = 0$$

This is unstable so $\theta = \pi$ (south pointing) is not stable

(iii) The non-zero couple component is Q_1 and in steady state $\theta = 0$ $\therefore \Omega_1 = 0$ $\therefore \dot{\Omega}_1 = 0$

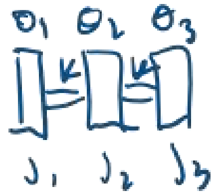
$$\therefore Q_1 = C\omega_3 \Omega_2$$

$$= C\omega\Omega \sin\lambda$$

[Aside: at the equator, $\lambda = 0$ $\therefore Q_1 = 0$ because the rotor stays aligned with the Earth's spin

at (near) the North pole, $\lambda = 90^\circ$ $\therefore Q_1 = C\omega\Omega$ which is the classic gyro couple as in Part IB mechanics, ie the couple needed to keep a gyro precessing at rate Ω and spin ω]

AC685 , Q3b



a) $q_1 = \theta_1 \quad q_2 = \theta_2 \quad q_3 = \theta_3$

$$T = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} J_3 \dot{\theta}_3^2$$

$$T = \frac{1}{2} \dot{\underline{q}}^T \underline{\underline{M}} \, \dot{\underline{q}}$$

$$\rightarrow \quad \underline{\underline{M}} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$$

b) $H(\theta, p_\theta) = T + V$ given (a)

$$V = \frac{1}{2} k (\theta_1 - \theta_2)^2 + \frac{1}{2} k (\theta_2 - \theta_3)^2$$

$$p_{\theta_i} = \frac{\partial T}{\partial \dot{\theta}_i} = J_i \dot{\theta}_i \Rightarrow \dot{\theta}_i = \frac{p_{\theta_i}}{J_i}$$

$$H = \frac{1}{2} \frac{p_{\theta_1}^2}{J_1} + \frac{1}{2} \frac{p_{\theta_2}^2}{J_2} + \frac{1}{2} \frac{p_{\theta_3}^2}{J_3} + \frac{1}{2} k (\theta_1 - \theta_2)^2 + \frac{1}{2} k (\theta_2 - \theta_3)^2$$

P1

c) $p_{TOT} = p_1 + p_2 + p_3 = J_1\dot{\theta}_1 + J_2\dot{\theta}_2 + J_3\dot{\theta}_3$

$$\{p_{TOT}, H\} = \underbrace{\frac{\partial p_{TOT}}{\partial \theta_1}}_{=0}\frac{\partial H}{\partial p_{\theta_1}} - \underbrace{\frac{\partial p_{TOT}}{\partial p_{\theta_1}}}_{=1}\underbrace{\frac{\partial H}{\partial \theta_1}}_{=k(\theta_1-\theta_2)} +$$

$$+ \underbrace{\frac{\partial p_{TOT}}{\partial \theta_2}}_{=0}\frac{\partial H}{\partial p_{\theta_2}} - \underbrace{\frac{\partial p_{TOT}}{\partial p_{\theta_2}}}_{=1}\underbrace{\frac{\partial H}{\partial \theta_2}}_{=k(2\theta_2-\theta_1-\theta_3)} +$$

$$+ \underbrace{\frac{\partial p_{TOT}}{\partial \theta_3}}_{=0}\frac{\partial H}{\partial p_{\theta_3}} - \underbrace{\frac{\partial p_{TOT}}{\partial p_{\theta_3}}}_{=1}\underbrace{\frac{\partial H}{\partial \theta_3}}_{=k(\theta_3-\theta_2)}$$

$$= k(\theta_1 - \theta_2) + k(2\theta_2 - \theta_1 - \theta_3) + k(\theta_3 - \theta_2) =$$

$$= 0$$

total momentum is conserved

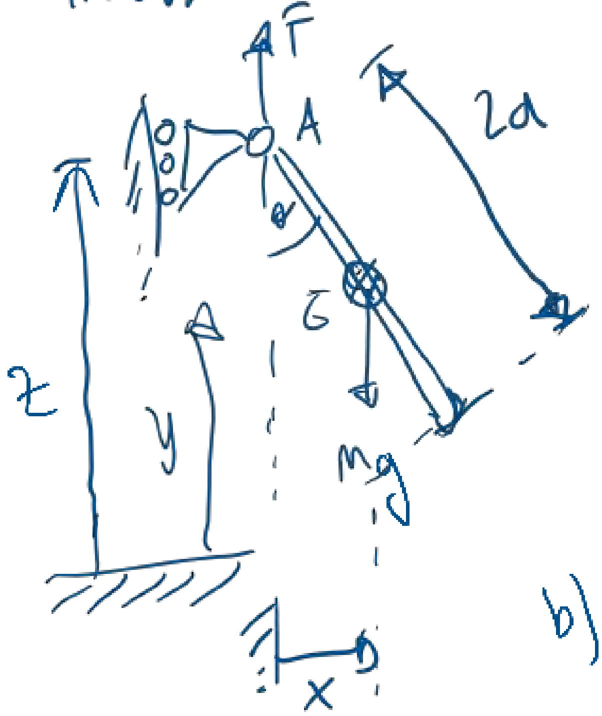
Justification: there are no external generalised torques → under an offset in angle (same in each q_i), the system is invariant.

AC685

Q4

$F = mg(1+b\cos\omega t)$

$q = z \qquad q_2 = \theta$



a) $\delta W = F\delta z$

$\delta W = mg(1+b\cos\omega t)\,\delta z$

$Q_z = mg(1+b\cos\omega t)$

$Q_\theta = 0$

b) $\dfrac{\partial V_{Qz}}{\partial z} = -Q_z \Rightarrow V_{Qz} = -\int Q_z\,dz = -z\,mg(1+b\cos\omega t)$

c) $L(z, \theta, \dot{z}, \dot{\theta}, t) = T - V + \underbrace{z\,mg(1+b\cos\omega t)}_{V_{Qz}}$

$V = mg(z - a\cos\theta)$

$T = \frac{1}{2}m\dot{x}_G^2 + \frac{1}{2}m\dot{y}_G^2 + \frac{1}{2}J_G\dot{\theta}^2$

$$\begin{bmatrix} x_G \\ y_G \end{bmatrix} = \begin{bmatrix} a\sin\theta \\ z - a\cos\theta \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \dot{x}_G \\ \dot{y}_G \end{bmatrix} = \begin{bmatrix} a\cos\theta\,\dot{\theta} \\ \dot{z} + a\sin\theta\,\dot{\theta} \end{bmatrix}$$

$$T = \frac{1}{2}m(a\cos\theta\,\dot{\theta})^2 + \frac{1}{2}m(\dot{z} + a\sin\theta\,\dot{\theta})^2 + \frac{1}{2}J_G\dot{\theta}^2$$

$$T = \frac{1}{2}\left(\frac{4Ma^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + \frac{1}{\cancel{2}}\cancel{2}m\dot{z}\,a\sin\theta\,\dot{\theta}$$

with $J_G = \frac{1}{12}m(2a)^2 = \frac{ma^2}{3}$

d i) for small θ $\quad \sin\theta \simeq \theta$

$$T = \frac{1}{2}\left(\frac{4ma^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}m\dot{z}^2 + m\dot{z}\dot{\theta}a\theta$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = \frac{4ma^2}{3}\dot{\theta} + m\dot{z}\theta a$$

$$p_z = \frac{\partial T}{\partial\dot{z}} = m\dot{z} + ma\cancel{\dot{\theta}\theta} = 0$$

d ii) For small θ, find Hamiltonian

$$H(p_z, p_\theta, z, \theta, t) = p_z\dot{z} + p_\theta\dot{\theta} - T + V \underbrace{- Q_z z}_{\text{potential energy carried by } Q_z}$$

$$\dot{z} = \frac{p_z}{m} \qquad \left(p_\theta - ma\theta\dot{z}\right) = \frac{4ma^2}{3}\dot{\theta}$$

$$\Rightarrow \dot{\theta} = \frac{p_\theta - \cancel{m}a\theta\frac{p_z}{\cancel{m}}}{\frac{4ma^2}{3}}$$

$$H = \frac{p_z^2}{M} + \frac{p_\theta\left(p_\theta - a\theta\, p_z\right)}{\frac{4ma^2}{3}} - \frac{1}{2}\frac{p_z^2}{M} - \frac{1}{2}\cancel{\left(\frac{4ma^2}{3}\right)}\frac{\left(p_\theta - a\theta\, p_z\right)^2}{\left(\frac{4ma^2}{3}\right)^{\cancel{2}}}$$

$$+ m\left(\frac{p_z}{M}\right)\theta a + mg\left(z - a\cos\theta\right)$$

$$- mg\left(1 + b\cos\omega t\right) z$$

additional time varying term caused by additional force

since $\frac{\partial V_z}{\partial z} = -Q_z$

e) (i) $H = T + V$

Since the T can be written in terms of the mass matrix, one would be inclined to say $T + V$ is a statement of energy conservation.

However, the Energy of the system is not conserved because of the time varying force leading to V_{ext}.

(ii) $$\frac{d}{dt} f(\underline{p}, \underline{q}, t) = \frac{\partial f}{\partial t} + \{f, H\}$$

$$\rightarrow \frac{d}{dt} H(p_z, p_\theta, z, \theta, t) = \frac{\partial H}{\partial t} + \{H, H\}$$

$\{H, H\} \neq 0$ H is not constant

$\Rightarrow$ Energy is not conserved.