Version AC/5

EGT2
ENGINEERING TRIPOS PART IIA

Wednesday 30 April 2025 9.30 to 11.10

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics and 3C6 Vibration datasheet 2023 (7 pages).
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A rigid framework of mass $4m$ comprises four thin bars each of mass m as shown in Fig. 1. Bars AB and CD are of length $2a$ and bars DE and BF are of length $\sqrt{2}a$. Using the (x, y, z) reference frame shown the coordinates of A, B, C, D, E and F are respectively $(-a, 0, 0)$,$(a, 0, 0)$,$(0, 0, -a)$,$(0, 0, a)$,$(0, \sqrt{2}a, a)$ and $(a, \sqrt{2}a, 0)$. The centre of mass G of the framework (not shown) is at $(\frac{a}{4}, \frac{\sqrt{2}a}{4}, \frac{a}{4})$.

(a) Find the moments of inertia I_{xx}, I_{yy} and I_{zz} of the framework at O and show that $I_{yy} = \frac{8}{3}ma^2$. [30%]

(b) Show that OG is a principal axis of the framework and find the principal moment of inertia about the axis OG. [40%]

It is given that principal moments of inertia of the framework at G are $\frac{5}{3}ma^2$, $\frac{5}{3}ma^2$ and $\frac{8}{3}ma^2$.

(c) Use the parallel axes theorem (or otherwise) to find the principal moments of inertia for the framework at O. Hence verify that the framework is *not* an "AAC" body at O. [30%]

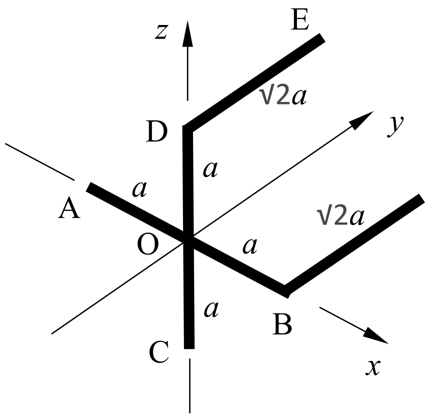


Fig. 1

2 A thin circular wheel of radius a, mass m and moments of inertia A, A, C with $C = ma^2$ is rolling without slip on a horizontal floor as shown in Fig 2. A reference frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (moving with but not fixed in the wheel) is shown in Fig. 2(b). The angular velocity of the reference frame $\boldsymbol{\Omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \Omega_3\mathbf{k}$ and the angular velocity of the wheel $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$. The angle from the vertical to the spin axis $\mathbf{k}$ is θ . The centre of the wheel G moves with velocity $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ as shown in Fig 2(c). The point of contact between wheel and floor is P.

(a) In steady straight-line rolling, as shown in Fig. 2(a), the forward speed of the wheel is V with $\theta = \frac{\pi}{2}$. Verify that $\alpha = 0$, $v_1 = v_3 = 0$, $v_2 = V$, $\omega_1 = \omega_2 = 0$, $\omega_3 = -V/a$ and $\Omega_3 = 0$. [10%]

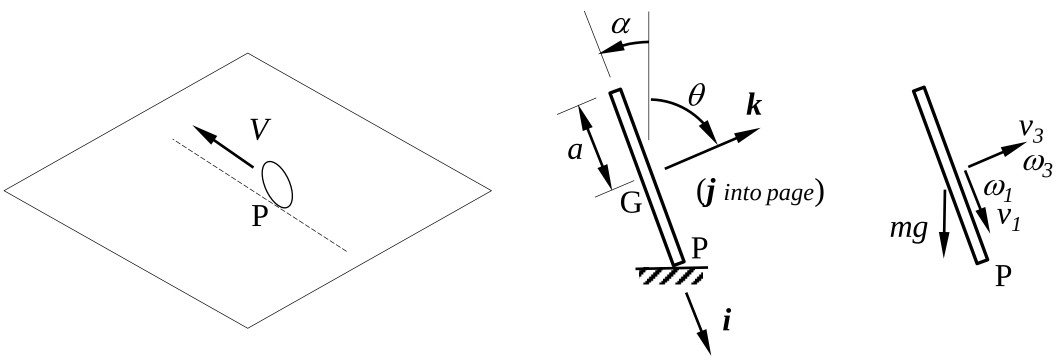
(b) Stability of steady rolling is examined by introducing a small perturbation α so that $\theta = \frac{\pi}{2} - \alpha$. Small-angle approximations can be used and assume fast spin, $\omega_3 >> \Omega_3$.

(i) Use the condition of no-slip at P to show that $v_1 = 0$, $v_2 = -a\omega_3$ and $v_3 = a\omega_2$, and $\dot{\boldsymbol{r}}_P \approx -a\omega_3\mathbf{j}$. [20%]

(ii) Show that the linear momentum of the wheel $\boldsymbol{p} = ma(-\omega_3\mathbf{j} + \omega_2\mathbf{k})$, the moment of momentum about P is $\boldsymbol{h}_P = \frac{1}{2}ma^2\left(\omega_1\mathbf{i} + 3\omega_2\mathbf{j} + 4\omega_3\mathbf{k}\right)$ and that the moment of external forces about P is $\boldsymbol{Q}_P = -mga\alpha\mathbf{j}$. [30%]

(iii) Use the datasheet expression for moment of momentum about a general point P to show that $\dot{\omega}_1 = -2\omega_2\omega_3$ and find a second-order differential equation governing small-amplitude wobbling of the wheel. [30%]

(iv) The angular velocity ω_2 is governed by $\ddot{\omega}_2 + \frac{2}{3}\left[4\omega_3^2 - \frac{g}{a}\right]\omega_2 = 0$. Find the minimum rolling speed V required for steady rolling in a straight line. [10%]



(a) (b) (c)

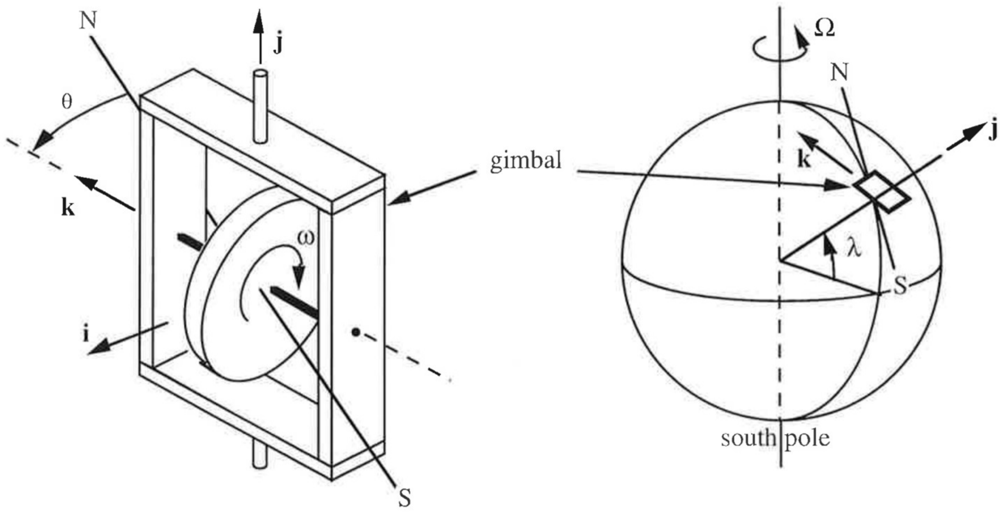
Fig. 2

3 (a) A simplified gyrocompass is shown in Fig. 3(a). The gimbal is free to turn through angle θ about its vertical axis $\mathbf{j}$. The rotor has principal moments of inertia A, A, C at its centre and it spins 'fast' at constant rate ω about its axis $\mathbf{k}$. The Earth spins at constant rate Ω and the instrument is situated at latitude λ as shown in Fig. 3(b). The components of the Earth's spin in the gimbal-mounted axis system $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are $-\Omega \cos\lambda \sin\theta$, $\Omega \sin\lambda$ and $\Omega \cos\lambda \cos\theta$ respectively.

(i) Use the Gyroscope Equations to obtain an equation of motion for the gyroscope (neglect friction and gimbal inertia). [30%]

(ii) Show that there is only one stable steady-state solution and it occurs when the gyrocompass aligns itself with true north. [10%]

(iii) What is the magnitude of the non-zero couple component in this steady-state? [10%]



(a) (b)

Fig. 3

 (cont.

(b) A simplified model of a jet engine is shown in Fig. 4. Three rigid discs with moments of inertia J_1, J_2 and J_3 are mounted on two light elastic shafts each of torsional stiffness k. The angles of rotations of the discs from their equilibrium positions are θ_1, θ_2 and θ_3 as shown.

(i) Evaluate the mass matrix for the system by using the concise form of the kinetic energy. [10%]

(ii) Evaluate the Hamiltonian of the system. [15%]

(iii) Use Poisson brackets to determine whether the total angular momentum of the system is conserved. Justify your answer on physical grounds. [25%]

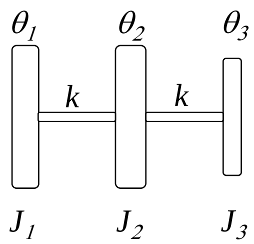


Fig. 4

4 A uniform rigid bar AB of mass m and length $2a$ is connected to a frictionless slider at A, whose position is described by the coordinate z, as shown in Fig. 5. The angle of the bar from the vertical is θ. A vertical force varying with time t, $F = mg(1 + b\cos(\omega t))$, is applied at A.

(a) Find the generalised forces Q_z and Q_θ. [10%]

(b) Find the expression of the potential energy generated by the generalised forces V_{Q_j} by using $\frac{\partial V_{Q_j}}{\partial q_j} = -Q_j$, where q_j is the j-th generalised coordinate. [10%]

(c) Evaluate the Lagrangian of the system, including V_{Q_j}, so that $L\left(z, \theta, \dot{z}, \dot{\theta}, t\right)$. [10%]

(d) For small theta:

(i) Find the generalised momenta p_z and p_θ; [10%]

(ii) Find the Hamiltonian $H\left(p_z, p_\theta, z, \theta, t\right)$ of this system including the V_{Q_j} obtained in part (b). [30%]

(e) Without further calculations:

(i) comment on the suitability of the expression $H = T + V$ for this system given the results obtained in part (d), and discuss if this expression can be used to provide a statement about the total energy of the system; [10%]

(ii) explain how the Poisson bracket can be used to investigate whether the energy of the system is conserved. Justify your answer on physical grounds. [20%]

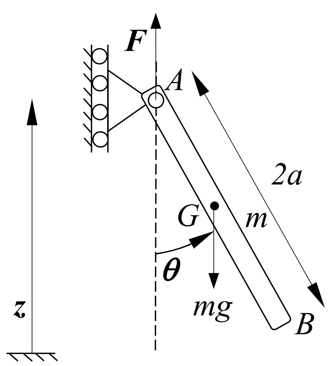


Fig. 5

END OF PAPER

Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\boldsymbol{v}_{\mathrm{G}}$, M is the total mass, $\boldsymbol{v}_{\mathrm{G}}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_{\mathrm{G}}$$
$$= \dot{\boldsymbol{h}}_{\mathrm{P}} + \dot{\boldsymbol{r}}_{\mathrm{P}} \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here $\boldsymbol{h}_{\mathrm{P}}$ and $\boldsymbol{h}_{\mathrm{G}}$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{P}}) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_{\mathrm{G}} + (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm = \boldsymbol{I}\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz \; dm \qquad E = \int zx \; dm \qquad F = \int xy \; dm$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= Q_3 \end{aligned}$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the"Gyroscope equations"):

$$\begin{aligned} A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot{\omega}_3 &= Q_3 \end{aligned}$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalised momenta p_i and the Hamiltonian $H(\boldsymbol{p},\boldsymbol{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad H(\boldsymbol{p},\boldsymbol{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\boldsymbol{p},\boldsymbol{q})$ must be expressed as a function of the generalised momenta and the generalised displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

Special case when the kinetic energy is expressible using a mass matrix $\boldsymbol{M}$:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M} \dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{p}^T \boldsymbol{M}^{-1} \boldsymbol{p} \qquad \text{and} \qquad H = T + V$$

(b) Extension topics

The total time derivative of some function $f(\boldsymbol{p},\boldsymbol{q},t)$ can be expressed in terms of the Poisson bracket $\{f,H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f,H\}, \qquad \{f,H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\boldsymbol{q},\boldsymbol{Q},t)$	$\boldsymbol{p} = \frac{\partial G_1}{\partial \boldsymbol{q}}$	$\boldsymbol{P} = -\frac{\partial G_1}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\boldsymbol{q},\boldsymbol{P},t)$	$\boldsymbol{p} = \frac{\partial G_2}{\partial \boldsymbol{q}}$	$\boldsymbol{Q} = \frac{\partial G_2}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\boldsymbol{p},\boldsymbol{Q},t)$	$\boldsymbol{q} = -\frac{\partial G_3}{\partial \boldsymbol{p}}$	$\boldsymbol{P} = -\frac{\partial G_3}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\boldsymbol{p},\boldsymbol{P},t)$	$\boldsymbol{q} = -\frac{\partial G_4}{\partial \boldsymbol{p}}$	$\boldsymbol{Q} = \frac{\partial G_4}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems

1. Equation of motion

The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

2. Kinetic Energy

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

3. Potential Energy

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

4. Natural frequencies and mode shapes

The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

5. Orthogonality and normalisation

$$\mathbf{u}^{(j)^T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)^T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

Continuous Systems

1. Equation of motion

The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

2. Kinetic Energy

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

See Section 3 for examples.

4. Natural frequencies and mode shapes

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_j q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\text{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)}u_k^{(n)}}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1,x_2,t) = \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1,x_2,t) \approx \sum_{n} \frac{u_n(x_1)u_n(x_2)}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2}\Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2}\Big[1 - e^{-\omega_n \zeta_n t}\cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2}\Big[1 - \cos\omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2}\Big[1 - e^{-\omega_n \zeta_n t}\cos\omega_n t\Big]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

3.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned}
&\cos\alpha\cosh\alpha + 1 = 0, && \alpha_1 = 1.8751\\
&\cos\alpha\cosh\alpha - 1 = 0, && \alpha_1 = 4.7300\\
&\tan\alpha - \tanh\alpha = 0, && \alpha_1 = 3.9266
\end{aligned}$$