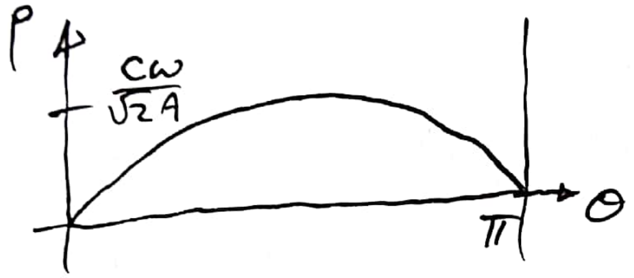
3C5 Dynamics Part IIA 2021

(Hugh Hunt, James Talbot)

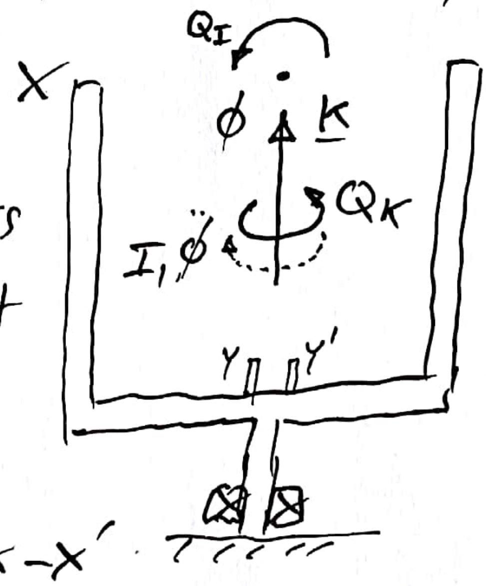
1(a) $p = \frac{C\omega}{A}\left[1 + \cot^2\theta + \operatorname{cosec}^2\theta\right]^{-\frac{1}{2}}$

$= \frac{C\omega}{A}\left[\operatorname{cosec}^2\theta + \operatorname{cosec}^2\theta\right]^{-\frac{1}{2}} = \frac{C\omega}{\sqrt{2}A\sin\theta}$



Note that $\cot\theta = 0$ for $\theta = \frac{\pi}{2}$ so the J_1 term vanishes. This is because rotation around $\theta = \frac{\pi}{2}$ is not associated with rotation of the assembly about $\underline{k}$

(b) Stand



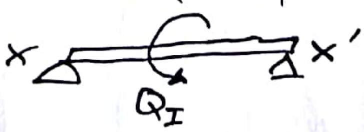
$X' \longrightarrow \underline{j}$

$\underline{I}$ out of page (i.e. in $\underline{k}$ direction for $\theta = \frac{\pi}{2}$)

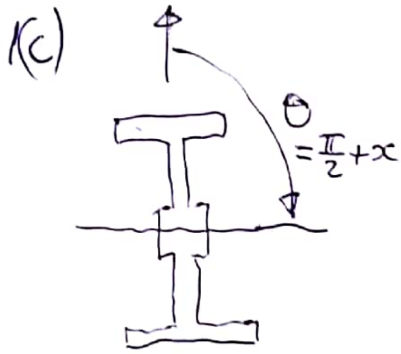
The assembly clicks in at X & X' (but it makes no difference to use Y & Y'). It is free to turn about X–X'. So this couple is zero. There is a d'Alembert inertia couple $I_1\ddot{\phi}$ about $\underline{k}$ and is equal to $I_1\ddot{\phi}$. So $Q_K\underline{k}$ acts on the stand

$Q_K = I_1\ddot{\phi}$

There is an as-yet unknown couple $Q_I\underline{I}$ acting on the stand because X–X' are spaced apart, as if simply supported



1(c)



rotor, gyro equations (fast spin)

$$Q_1 = A\dot{\Omega}_1 - (\cancel{A\Omega_3} - C\omega_3)\Omega_2 \qquad (1)$$

$$Q_2 = A\dot{\Omega}_2 + (\cancel{A\Omega_3} - C\omega_3)\Omega_1 \qquad (2)$$

$$Q_3 = C\dot{\omega}_3 \quad (\text{constant}) \quad (Q_3 = 0) \qquad (3)$$

$$\Omega_1 = -\dot{\phi}\sin\theta \simeq -\dot{\phi}$$

$$\Omega_2 = \dot{\theta} = \dot{x}$$

$$\Omega_3 = \dot{\phi}\cos\theta \simeq -\dot{\phi}x = 0$$

1(d) linearize about $\theta_0 = \frac{\pi}{2}$

put $\theta = \frac{\pi}{2} + x$ (x small)

$\sin\theta \simeq 1$

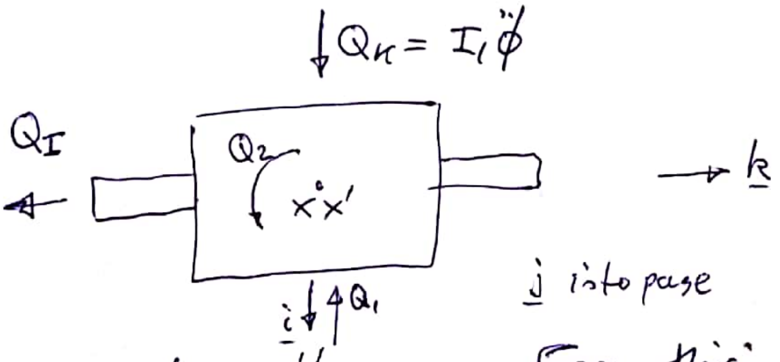
$\cos\theta \simeq -x$

Note that, for small oscillations, x & $\dot{\phi}$ are small so ignore terms like x^2, $x\dot{\phi}$, $\dot{\phi}^2$ etc

So (1)

$$\boxed{\begin{aligned} Q_1 &\simeq -A\ddot{\phi} + C\omega_3\dot{x} \qquad (1)^a \\ Q_2 &\simeq A\ddot{x} + C\omega_3\dot{\phi} \qquad (2)^a \end{aligned}}$$

& (2)



Assembly

Q_I is in opposite direction to Q_I on stand (action & reaction)

From this, $Q_1 = I_1\ddot{\phi}$ & $Q_2 = 0$

$\therefore$ $(1)^a$ becomes $(I_1 + A)\ddot{\phi} = C\omega_3\dot{x}$

integrate $\therefore$ ~~of~~ $\cancel{I_1+A}$ $\dot{\phi} = \frac{C\omega_3}{I_1 + A}x + \text{const}$

$(1)^b$ becomes $0 = A\ddot{x} + \frac{(C\omega_3)^2}{I_1 + A}x + \text{const}$

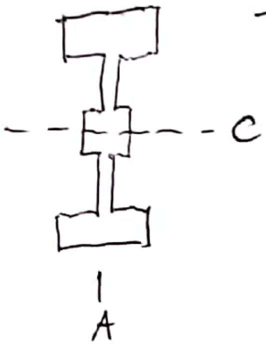
This gives SHM at frequency $p^* = \frac{C\omega_3}{A}\left(1 + \frac{I_1}{A}\right)^{-\frac{1}{2}}$

which is correct for $\theta_0 = \frac{\pi}{2}$ & $J_1 = 0$

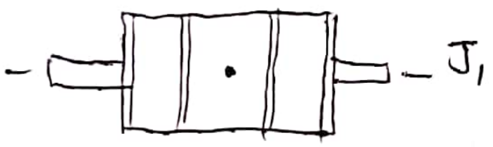
Q2 More generally (not expected)

$$p = \frac{C\omega}{A}\left[1 + \frac{J_1}{A}\cot^2\theta_0 + \frac{I_1}{A}\operatorname{cosec}^2\theta_0\right]^{-\frac{1}{2}}$$

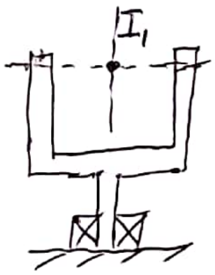
This is the nutation rate for the 3C5 lab gyro in its gimbal frame.



The rotor is AAC in assembly-fixed axes $\underline{i}\,\underline{j}\,\underline{k}$
Note that "A" includes the moment of inertia of the gyro assembly

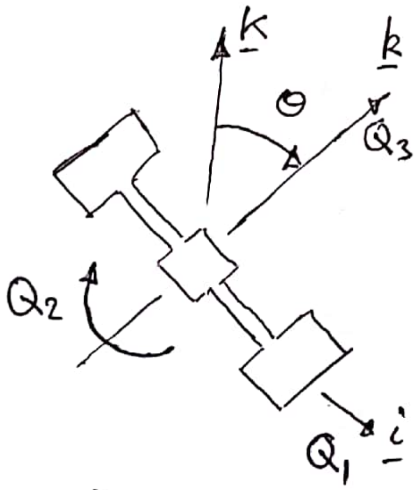


The gyro assembly has moment of inertia J_1 about its axis $\underline{k}$

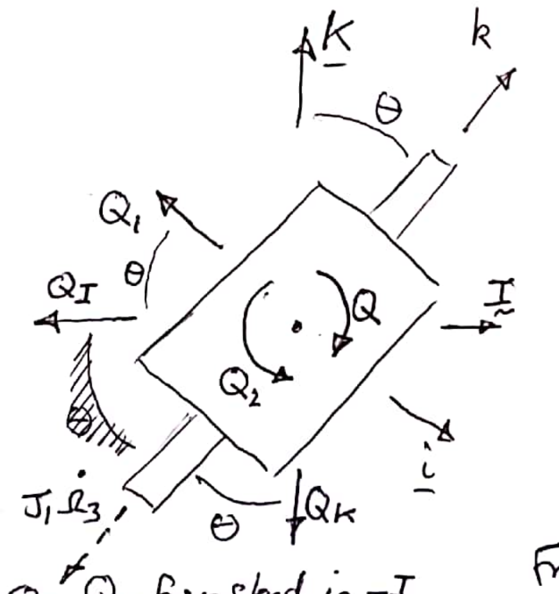


The support stand has moment of inertia I_1 about the vertical axis $\underset{\sim}{K}$

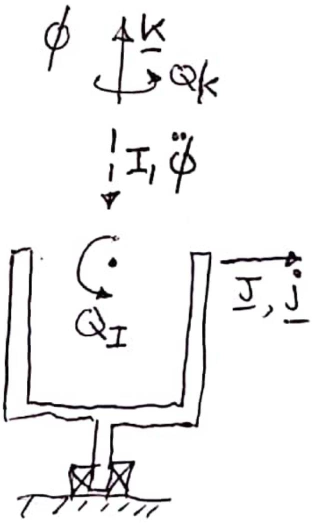
Angle of tilt is θ, Spin is $\omega_3\,\underline{k}$



$\underline{j}$ into page



Q_I, Q_K from stand is $-\underset{\sim}{I}$ & $-\underline{k}$ directions



Frictionless about $\underline{j}$

Couples Q_1, Q_2 & Q_3 act on the rotor
$\therefore$ equal & opposite on assembly

External couple $Q\,\underline{j}$

Couples Q_I, Q_K about $\underline{I}$ axis & $\underline{K}$ axis ($\underline{I}$ out of page) acting on stand

Gyro equations for rotor

Euler angles

$\Omega_1 = -\dot\phi \sin\theta$

$\Omega_2 = \dot\theta$

$\Omega_3 = \dot\phi \cos\theta$

$\omega_3 = \Omega_3 + \dot\psi$

$\underline{i}$ ① $Q_1 = A\dot\Omega_1 - (A\Omega_3 - C\omega_3)\Omega_2$

$\underline{j}$ ② $Q_2 = A\dot\Omega_2 + (A\Omega_3 - C\omega_3)\Omega_1$

$\underline{k}$ ③ $Q_3 = C\dot\omega_3 = 0$ (frictionless about $\underline{k}$)

$\therefore \omega_3 = \text{constant}$

Inertia couples $-I_1\ddot\phi\,\underline{K}$ and $-J_1\dot\Omega_3\,\underline{k}$ act on ~~stand~~ and assembly

so for stand $Q_K = I_1\ddot\phi$ $\quad|\underline{K}$ ⑦

and for assembly $Q_I\cos\theta + Q_1 - Q_K\sin\theta = 0$ $\quad\underline{i}$ ④

$Q_2 - Q = 0$ $\quad\underline{j}$ ⑤

$Q_I\sin\theta + Q_K\cos\theta + J_1\dot\Omega_3 = 0$ $\quad\underline{k}$ ⑥

In steady state precession due to Q

$\theta = \theta_0 = \text{const}$

$\dot\phi = \dot\phi_0 = \text{const}$

$\therefore \Omega_1 = -\dot\phi_0\sin\theta_0 = \text{const}$

$\Omega_2 = \dot\theta = 0$

$\Omega_3 = \dot\phi_0\cos\theta_0 = \text{const}$

Gyro equation ② is the only meaningful one

$\therefore Q_2 = -(A\Omega_3 - C\omega_3)\dot\phi_0\sin\theta_0$

and for fast spin $\boxed{Q_2 \simeq C\omega_3\dot\phi_0\sin\theta_0}$

and with ⑤ $Q = C\omega_3\dot\phi_0\sin\theta_0$ ⑧

Now for nutation perturb the motion

$\theta = \theta_0 + x \qquad \dot\theta = \dot x \qquad \ddot\theta = \ddot x \qquad x, y$ small

$\dot\phi = \dot\phi_0 + y \qquad\qquad \ddot\phi = \dot y$

$\Omega_1 = -(\dot\phi_0 + y)\sin(\theta_0 + x) = -(\dot\phi_0 + y)(\sin\theta_0\cos x + \cos\theta_0\sin x)$

and $\sin x \approx x \qquad \cos x \approx 1$

$$\therefore \quad \Omega_1 \simeq -(\dot\phi_0 + y)(\sin\theta_0 + x\cos\theta_0)$$

$$\approx -\dot\phi_0 \sin\theta_0 - y\sin\theta_0 - x\cos\theta_0\dot\phi_0$$

$$\& \quad \Omega_2 \simeq \dot x$$

$$\& \quad \Omega_3 \simeq (\dot\phi_0 + y)\cos(\theta_0 + x)$$

$$\simeq (\dot\phi_0 + y)(\cos\theta_0\cos x - \sin\theta_0 \sin x)$$

$$= (\dot\phi_0 + y)(\cos\theta_0 - x\sin\theta_0)$$

$$\simeq \dot\phi_0 \cos\theta_0 + y\cos\theta_0 - x\sin\theta_0\,\dot\phi_0$$

$$\dot\Omega_1 \simeq -\dot y \sin\theta_0 - \dot x\cos\theta_0\dot\phi_0$$

$$\dot\Omega_2 = \ddot x$$

$$\dot\Omega_3 = \dot y\cos\theta_0 - \dot x\sin\theta_0\dot\phi_0$$

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④ $\therefore \quad Q_1 = Q_K \sin\theta - Q_I\cos\theta$

⑦ $\quad Q_K = I_1\ddot\phi = I_1\dot y$

⑥ $\quad Q_I \sin\theta = -Q_K\cos\theta - J_1\dot\Omega_3$

$$\therefore Q_I \sin(\theta_0 + x) = -I_1\dot y\cos(\theta_0+x) - J_1(\dot y\cos\theta_0 - \dot x\sin\theta_0\dot\phi_0)$$

$$\therefore Q_I(\sin\theta_0 + x\cos\theta_0) = -I_1\dot y(\cos\theta_0 - x\sin\theta_0) - J_1(\dot y\cos\theta_0 - \dot x\sin\theta_0\dot\phi_0)$$

$$\therefore Q_I \sin\theta_0(1 + x\cot\theta_0) = -I_1\cos\theta_0\dot y - J_1\cos\theta_0\dot y + J_1\sin\theta_0\dot\phi_0\dot x$$

(note $\dot y x$ is verysmall)

$$\therefore Q_I = \frac{(1 - x\cot\theta_0)}{\sin\theta_0}\left(J_1\sin\theta_0\dot\phi_0\dot x - (I_1 + J_1)\cos\theta_0\dot y\right)$$

$$= \frac{1}{\sin\theta_0}\left(J_1\sin\theta_0\dot\phi_0\dot x - (I_1+J_1)\cos\theta_0\dot y\right)$$

$$\therefore Q_1 = I_1\dot y\sin(\theta_0 + x) - \frac{\cos(\theta_0+x)}{\sin\theta_0}\left(J_1\sin\theta_0\dot\phi_0\dot x - (I_1+J_1)\cos\theta_0\dot y\right)$$

$$= I_1\dot y(\sin\theta_0 + x\cos\theta_0) - \frac{\cos\theta_0 - x\sin\theta_0}{\sin\theta_0}\left(\qquad \right)$$

$$= I_1\dot y\sin\theta_0 - \cot\theta_0\left(J_1\sin\theta_0\dot\phi_0\dot x - (I_1+J_1)\cos\theta_0\dot y\right)$$

$$\therefore\ Q_1 = I_1 \dot{y} \sin\theta_0 - J_1 \cos\theta_0 \dot{\phi}_0 \dot{x} + (I_1 + J_1)\frac{\cos^2\theta_0}{\sin\theta_0}\dot{y}$$

note $\sin\theta_0 + \dfrac{\cos^2\theta_0}{\sin\theta_0} = \dfrac{\sin^2\theta_0 + \cos^2\theta_0}{\sin\theta_0} = \operatorname{cosec}\theta_0$

$$\therefore\ Q_1 = -J_1 \cos\theta_0 \dot{\phi}_0 \dot{x} + \left(J_1 \cos\theta_0 \cot\theta_0 + I_1 \operatorname{cosec}\theta_0\right)\dot{y}$$

and ① $Q_1 = -A(\dot{y}\sin\theta_0 + \dot{x}\cos\theta_0 \dot{\phi}_0) + C\omega_3 \dot{x}$ (fast spin)

$= -A\sin\theta_0 \dot{y} + (C\omega_3 - \cancel{A\cos\theta_0 \dot{\phi}_0})\dot{x}$ (fast spin)

$$\therefore\ \left(C\omega_3 \cancel{- A\cos\theta_0 \dot{\phi}_0} + \cancel{J_1 \cos\theta_0 \dot{\phi}_0}\right)\dot{x} = \left(J_1 \cos\theta_0 \cot\theta_0 + I_1 \operatorname{cosec}\theta_0 + A\sin\theta_0\right)\dot{y}$$

② $Q = A\ddot{x} + C\omega_3(\dot{\phi}_0 \sin\theta_0 + y\sin\theta_0 - x\cos\theta_0 \dot{\phi}_0)$

and note steady state response from ⑧

$$\therefore\ A\ddot{x} + C\omega_3 \sin\theta_0\, y - C\omega_3 \cos\theta_0 \dot{\phi}_0\, x = 0$$

fast spin

$$\therefore\ A\ddot{x} + C\omega_3\left(\frac{\sin\theta_0\left(C\omega_3 \cancel{- A\cos\theta_0 \dot{\phi}_0 + J_1\cos\theta_0 \dot{\phi}_0}\right)}{J_1\cos\theta_0\cot\theta_0 + I_1\operatorname{cosec}\theta_0 + A\sin\theta_0} \cancel{- \cos\theta_0 \dot{\phi}_0}\right)x = \text{Const}$$

This is SHM at frequency p

$$\ddot{x} + \underbrace{\left(\frac{C\omega_3}{A}\right)^2 \frac{A\sin\theta_0}{J_1\cos\theta_0\cot\theta_0 + I_1\operatorname{cosec}\theta_0 + A\sin\theta_0}}_{p^2}\, x = \text{Const}$$

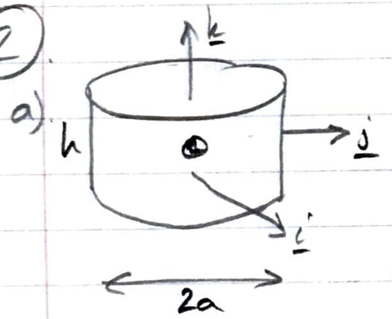
$$p^2 = \left(\frac{C\omega_3}{A}\right)^2\left[\frac{J_1}{A}\cot^2\theta_0 + \frac{I_1}{A}\operatorname{cosec}^2\theta_0 + 1\right]^{-1}$$

$$\therefore\ p = \frac{C\omega_3}{A}\left[1 + \frac{J_1}{A}\cot^2\theta_0 + \frac{I_1}{A}\operatorname{cosec}^2\theta_0\right] \quad \underline{\text{QED}}$$

2021

(2)

a)



About G: $I_{zz} = \frac{1}{2} m a^2$

$$I_{xx} = I_{yy} = \left(\frac{a^2}{4} + \frac{h^2}{12}\right) m$$

$\therefore$ 'AAA' $\Rightarrow \frac{a^2}{4} + \frac{h^2}{12} = \frac{a^2}{2} \quad \therefore h = a\sqrt{3}$

b). No slip at P $\Rightarrow \underline{\omega} \times (a\underline{i} - \frac{h}{2}\underline{k}) = 0$.

(i)

ie. $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ a & 0 & -h/2 \end{vmatrix} = \underline{i}\left(-\frac{h}{2}\omega_2\right) - \underline{j}\left(-\frac{h}{2}\omega_1 - a\omega_3\right) + \underline{k}\left(a\omega_2\right) = 0$

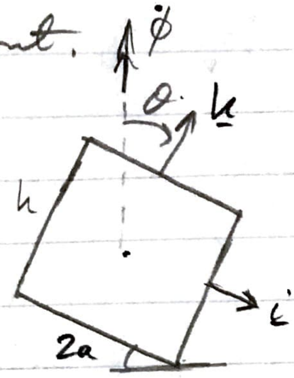
$$\Rightarrow -\frac{h}{2}\omega_2 \underline{i} + \left(\frac{h}{2}\omega_1 + a\omega_3\right)\underline{j} - a\omega_2 \underline{k} = 0$$

$$\therefore \omega_2 = 0 \text{ and } \omega_3 = -\frac{h}{2a}\omega_1$$

(ii)

$A - C = \frac{15}{4} m a^2 \quad \therefore \dot{\psi}^2 \geq 8 \cancel{m} g \cdot \frac{3\sqrt{3}}{4} a \cdot \frac{15}{4} \cancel{m} a^2 \cdot \frac{36}{\cancel{m^2} a^4} = 810\sqrt{3}\,\frac{g}{a} \Rightarrow \dot{\psi} \geq 37.46\sqrt{\frac{g}{a}}$

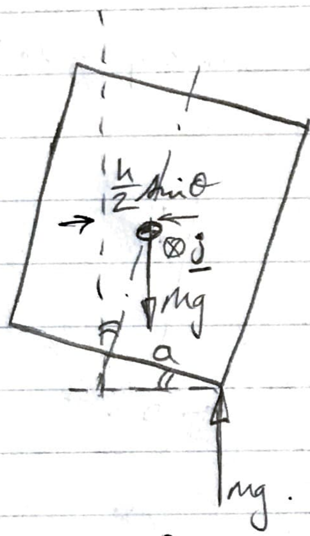
② b). Cont.



$\omega_1 = \Omega_1 = -\dot\phi \sin\theta$
$\omega_2 = \Omega_2 = \dot\theta$
$\Omega_3 = \dot\phi \cos\theta$

$\therefore$ no slip $\Rightarrow$ $\omega_3 = -\frac{h}{2a}(-\dot\phi\sin\theta)$
$= \frac{h}{2a}\dot\phi\sin\theta.$

(ii).



$Q_2 = -mg(a\cos\theta - \frac{h}{2}\sin\theta)$

$= mg(\frac{h}{2}\sin\theta - a\cos\theta)$

(iii) $Q_2 = A\dot\Omega_2 + (A\Omega_3 - C\omega_3)\Omega_1$, $\dot\Omega_2 = 0$ and 'AAA' body

$\therefore mg\left(\frac{h}{2}\sin\theta - a\cos\theta\right) = \left(A\dot\phi\cos\theta - A.\frac{h}{2a}\dot\phi\sin\theta\right)\left(-\dot\phi\sin\theta\right)$

$mg\frac{h}{2}\sin\theta - mga\cos\theta = -A\dot\phi^2\sin\theta\cos\theta + \frac{Ah}{2a}\dot\phi^2\sin^2\theta$

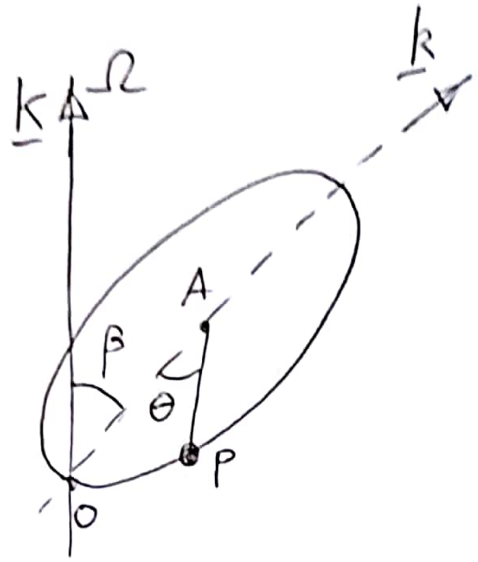
$mgah\sin\theta - 2mga^2\cos\theta = -2aA\dot\phi^2\sin\theta\cos\theta + Ah\dot\phi^2\sin^2\theta$

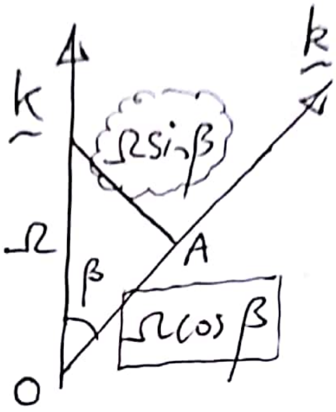
$mga(\cancel{h\sin\theta - 2a\cos\theta}) = A\dot\phi^2\sin\theta(\cancel{h\sin\theta - 2a\cos\theta})$

$\Rightarrow \dot\phi^2 = \frac{mga}{A\sin\theta}$

But $A = \frac{1}{2}ma^2$ $\therefore$ $\dot\phi^2 = \frac{2g}{a\sin\theta}$

3(a)

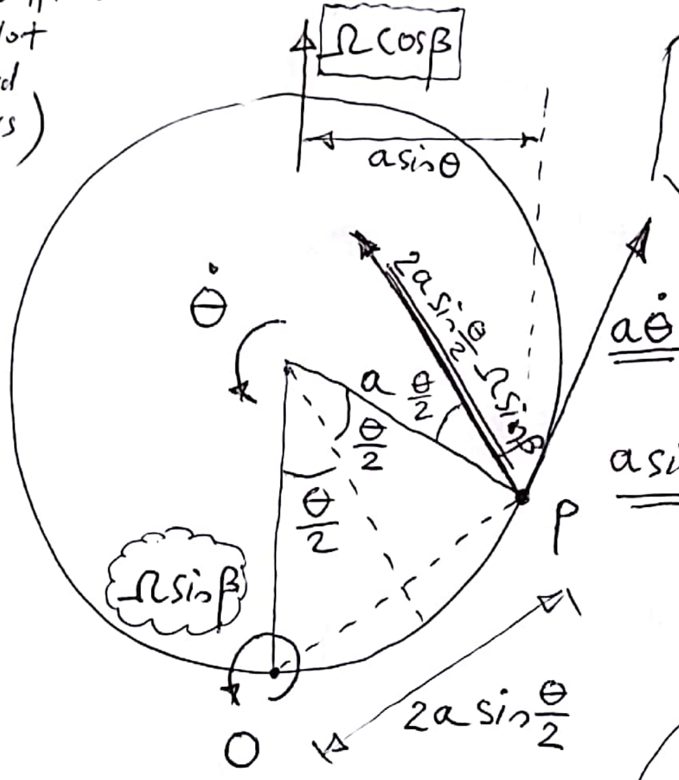




resolve Ω into & out of plane of ring

(Note: This solution is for a complete treatment. Not all is required for full marks (eg page 4))

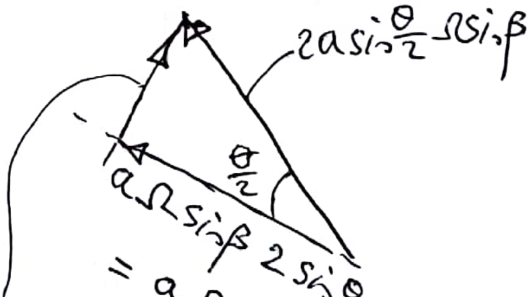
Double-underlined are the velocity components of P due to Ω & $\dot\theta$



$\underline{\underline{a\dot\theta}}$

$\underline{\underline{a\sin\theta\,\Omega\cos\beta}}$ into page

True view of ring



$a\Omega\sin\beta\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
$= a\Omega\sin\beta\,(\sin\theta)$

$a\Omega\sin\beta\, 2\sin\frac{\theta}{2}\sin\frac{\theta}{2}$
$= a\Omega\sin\beta(1-\cos\theta)$

The three velocity components of P in an orthogonal set are

$a[\dot\theta + \Omega\sin\beta(1-\cos\theta)]$ ↗

$a\Omega\sin\beta\sin\theta$ ↖

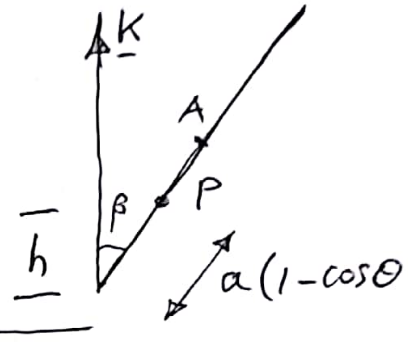
$a\Omega\cos\beta\sin\theta$ • into page

$$\therefore \quad T = \frac{1}{2} m a^2 \left[\Omega^2 \sin^2\beta \sin^2\theta + \Omega^2 \cos^2\beta \sin^2\theta + \left(\dot\theta + \Omega \sin\beta (1-\cos\theta)\right)^2 \right]$$

$$= \frac{1}{2} m a^2 \left[\Omega^2 \sin^2\theta + \left(\dot\theta + \Omega \sin\beta(1-\cos\theta)\right)^2 \right]$$

$$\& \quad V = mgh$$

$$= mga(1-\cos\theta)\cos\beta$$



$$\frac{\partial T}{\partial \dot\theta} = ma^2\left(\dot\theta + \Omega \sin\beta(1-\cos\theta)\right)$$

$$\frac{\partial T}{\partial \theta} = ma^2\left(\Omega^2 \sin\theta\cos\theta + \left(\dot\theta + \Omega\sin\beta(1-\cos\theta)\right)\Omega\sin\beta\sin\theta\right)$$

$$\cancel{= ma^2(\Omega^2 \sin\theta \cos\theta +}$$

$$= ma^2 \sin\theta\left(\Omega^2\cos\theta + \dot\theta\Omega\sin\beta + \Omega^2\sin^2\beta(1-\cos\theta)\right)$$

$$= ma^2\sin\theta\left(\Omega^2[\cos\theta(1-\sin^2\beta) + \sin^2\beta] + \dot\theta\Omega\sin\beta\right]$$

$$= ma^2\Omega\sin\theta\left(\Omega(\cos\theta\cos^2\beta + \sin^2\beta) + \dot\theta\sin\beta\right)$$

$$\frac{\partial V}{\partial \theta} = mga\sin\theta\cos\beta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0 \qquad (\text{Lagrange})$$

$$\therefore \quad \ddot\theta + \cancel{\Omega\sin\beta\,\dot\theta\sin\theta} - \cancel{\Omega\sin\beta\,\dot\theta\sin\theta} - \Omega^2\sin\theta(\cos\theta\cos^2\beta + \sin^2\beta) + \frac{g}{a}\sin\theta\cos\beta = 0$$

$$\therefore \quad \ddot\theta + \left[\frac{g}{a}\cos\beta - \Omega^2(\cos\theta\cos^2\beta + \sin^2\beta)\right]\sin\theta = 0$$

(NB put $\beta = 0$

$$\therefore \ddot{\theta} + \left[\frac{g}{a} - \Omega^2 \cos\theta\right] \sin\theta = 0$$

as in IB Mechanics examples paper)

For steady state solutions put $\ddot{\theta} = 0$

$$\therefore \left[\frac{g}{a}\cos\beta - \Omega^2\left(\cos\theta\cos^2\beta + \sin^2\beta\right)\right]\sin\theta = 0$$

$\therefore \sin\theta = 0$ ie $\boxed{\theta = 0}$ or (1) $\boxed{\theta = \pi}$ (2)

or $\frac{g}{a}\cos\beta - \Omega^2\left(\cos\theta\cos^2\beta + \sin^2\beta\right) = 0$

$$\therefore \cos\theta = \left(\frac{g\cos\beta}{a\Omega^2} - \sin^2\beta\right)\frac{1}{\cos^2\beta}$$

$$\boxed{\cos\theta = \frac{g}{a\Omega^2\cos\beta} - \tan^2\beta}$$

(3) only exists if $\cos\theta \leq 1$

$\therefore \frac{g}{a\Omega^2\cos\beta} - \tan^2\beta \leq 1$ $\therefore \frac{g\cos\beta}{a\Omega^2} \leq 1$

Check for stability

(1) $\theta_0 = 0$ $\therefore$ put θ small $\sin\theta \simeq \theta$, $\cos\theta \simeq 1$

$$\ddot{\theta} + \left[\frac{g}{a}\cos\beta - \Omega^2\left(\cos^2\beta + \sin^2\beta\right)\right]\theta = 0$$

$$\therefore \ddot{\theta} + \left[\frac{g}{a}\cos\beta - \Omega^2\right]\theta = 0$$

only stable if $\frac{g\cos\beta}{a\Omega^2} > 1$

unstable otherwise

note that solution (3) exists for $\frac{g\cos\beta}{a\Omega^2} \leq 1$

or $\Omega^2 \geq \frac{g\cos\beta}{a}$

② $\Theta_0 = \pi$ put $\Theta = \pi + x$ with small x

$$\sin\Theta = -\sin x \simeq -x$$

$$\cos\Theta \simeq -1$$

$$\ddot{x} + \left[\frac{g}{a}\cos\beta - \Omega^2\left(-\cos^2\beta + \sin^2\beta\right)\right](-x) = 0$$

$$\therefore \ddot{x} - \left[\frac{g}{a}\cos\beta + \cos 2\beta\right] x = 0$$

always unstable

③ $\cos\Theta_0 = \frac{g}{a\Omega^2\cos\beta} - \tan^2\beta$

put $\Theta = \Theta_0 + x$

$$\sin\Theta = \sin\Theta_0 \cos x + \cos\Theta_0 \sin x$$
$$= \sin\Theta_0 + x\cos\Theta_0$$
$$\cos\Theta = \cos\Theta_0 \cos x - \sin\Theta_0 \sin x$$
$$= \cos\Theta_0 - x\sin\Theta_0$$

$$\therefore \ddot{x} + \left[\frac{g}{a}\cos\beta - \Omega^2\left[\left(\cos\Theta_0 - x\sin\Theta_0\right)\cos^2\beta + \sin^2\beta\right]\right]\left[\sin\Theta_0 + x\cos\Theta_0\right] = 0$$

$$\therefore \ddot{x} + \left[\underbrace{\frac{g}{a}\cos\beta - \Omega^2\left(\cos\Theta_0\cos^2\beta + \sin^2\beta\right)}_{=0}\right]\left[\sin\Theta_0 + x\cos\Theta_0\right] + \Omega^2 x \sin\Theta_0 \cos^2\beta \sin\Theta_0 = 0$$

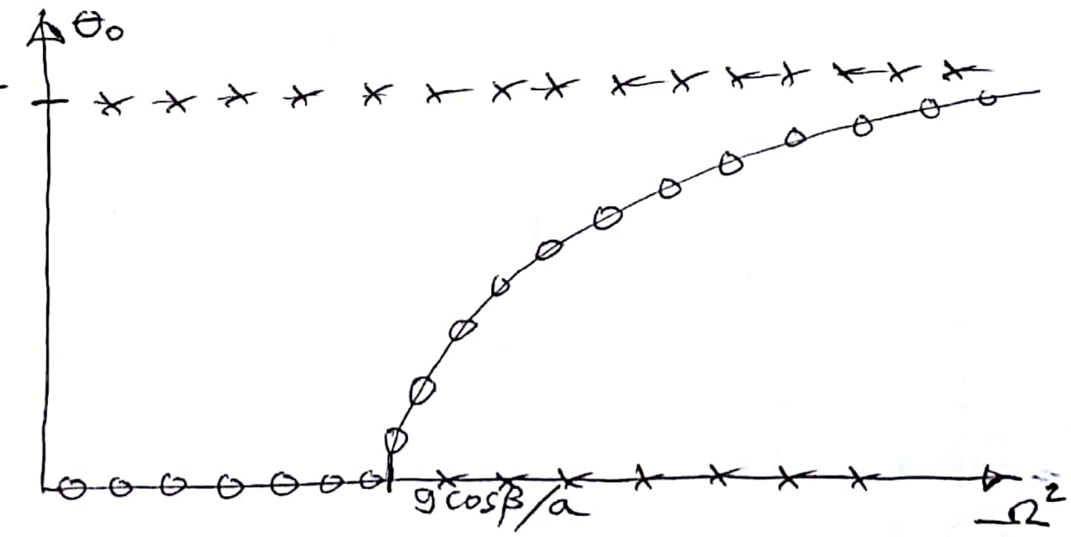
$$\therefore \ddot{x} + \Omega^2 \sin^2\Theta_0 \cos^2\beta\, x = 0$$

always stable

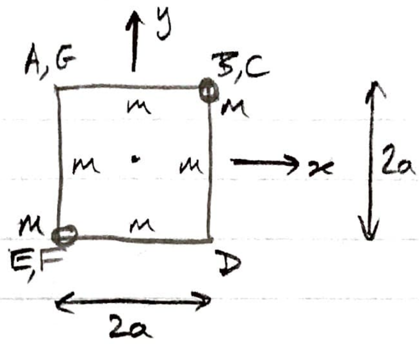
In summary

x = unstable

o = stable



4) a). Topview (down z-axis)



$$I_{zz} = 4\left(\tfrac{1}{3}ma^2 + ma^2\right) + 2\left(m(a\sqrt{2})^2\right)$$

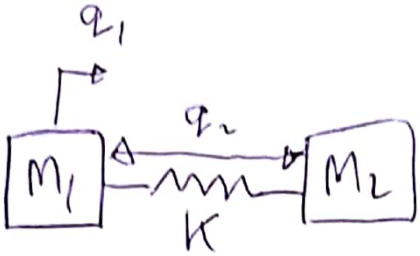
'rods' 'equivalent masses'

$$= \frac{16}{3}ma^2 + 4ma^2$$

$$= \frac{28}{3}ma^2$$

$I_{xy} = \int xy \, dm = 2ma^2$ (only equivalent masses contribute).

4b)



$$T = \tfrac{1}{2} m_1 \dot{q}_1^2 + \tfrac{1}{2} m_2 (\dot{q}_1 + \dot{q}_2)^2$$
$$V = \tfrac{1}{2} k q_2^2$$

$$p_1 = \frac{\partial T}{\partial \dot{q}_1} = m_1 \dot{q}_1 + m_2 (\dot{q}_1 + \dot{q}_2)$$

$$p_2 = \frac{\partial T}{\partial \dot{q}_2} = m_2 (\dot{q}_1 + \dot{q}_2) \qquad \therefore (\dot{q}_1 + \dot{q}_2) = \frac{p_2}{m_2}$$

$$\dot{q}_2 = \frac{p_2}{m_2} - \frac{p_1 - p_2}{m_1}$$

$$p_1 = m_1 \dot{q}_1 + p_2 \qquad \therefore \quad \dot{q}_1 = \frac{p_1 - p_2}{m_1}$$

$$\therefore T = \tfrac{1}{2} m_1 \left(\frac{p_1 - p_2}{m_1}\right)^2 + \tfrac{1}{2} m_2 \left(\frac{p_2}{m_2}\right)^2$$
$$= \frac{1}{2} \frac{(p_1 - p_2)^2}{m_1} + \frac{1}{2} \frac{p_2^2}{m_2}$$

$$H = p_1 \frac{p_1 - p_2}{m_1} + p_2 \left(\frac{p_2}{m_2} - \frac{p_1 - p_2}{m_1}\right) - \frac{1}{2} \frac{(p_1 - p_2)^2}{m_1} - \frac{1}{2} \frac{p_2^2}{m_2} + \tfrac{1}{2} k q_2^2$$

$$= \frac{(p_1 - p_2)(p_1 - p_2)}{m_1} + \frac{p_2^2}{m_2} - \frac{1}{2} \frac{p_2^2}{m_2} - \frac{1}{2} \frac{(p_1 - p_2)^2}{m_1} + \tfrac{1}{2} k q_2^2$$

$$= \cancel{\frac{(p_1^2 - p_2)^2}{m_1}} \cancel{+ \frac{p_2^2}{m_2}} + \frac{1}{2} \frac{p_2^2}{m_2} + \frac{1}{2} \frac{(p_1 - p_2)^2}{m_1} + \tfrac{1}{2} k q_2^2$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = \cancel{\frac{2p_1}{m_1}} \; \frac{p_1 - p_2}{m_1} = \frac{p_1 - p_2}{m_1} \quad ✓$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = \cancel{\frac{2p_2}{m_2}} \; \frac{p_2}{m_2} - \frac{p_1 - p_2}{m_1} = \cancel{\frac{2p_2}{m_2} - \frac{p_1}{m_1}} \quad ✓$$

4b cont.

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = 0$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -Kq_2$$

$$H = \frac{1}{2}[P_1 \quad P_2]\begin{bmatrix} M_1+M_2 & \cancel{M_1} M_2 \\ M_2 & M_2 \end{bmatrix}^{-1}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \frac{1}{2}Kq_2^2$$

$$= \frac{1}{2\Delta}(P_1 \quad P_2)\begin{pmatrix} \overset{M_2}{\cancel{M_1+M_2}} & -M_2 \\ -M_2 & M_1+M_2 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} + \frac{1}{2}Kq_2^2$$

$$= \frac{1}{2\Delta}(P_1 \quad P_2)\begin{pmatrix} M_2P_1 - M_2P_2 \\ -M_2P_1 + (M_1+M_2)P_2 \end{pmatrix} + \dots$$

$$= \frac{1}{2\Delta}\Big(P_1(M_2P_1 - M_2P_2) + P_2(-M_2P_1 + (M_1+M_2)P_2)\Big)$$

$$= \frac{1}{2\cancel{\Delta}} \frac{-2P_1M_2P_2 + P_1^2M_2 + P_2^2(M_1+M_2)}{\underbrace{(M_1+M_2)M_2 - M_2^2}_{= M_1M_2}}$$

$$= \cancel{\frac{1}{2}} = \frac{1}{2}\frac{(P_1-P_2)^2}{M_1} + \frac{1}{2}\frac{P_2^2}{M_2}$$