Version JPT/6

EGT2
ENGINEERING TRIPOS PART IIA

Thursday 29 April 2021   1.30 to 3.10

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet and at the top of each answer sheet.*

STATIONERY REQUIREMENTS
Write on single-sided paper.

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed.
Attachment: 3C5 Dynamics and 3C6 Vibration data sheet (7 pages).
You are allowed access to the electronic version of the Engineering Data Books.

10 minutes reading time is allowed for this paper at the start of the exam.

The time taken for scanning/uploading answers is 15 minutes.

Your script is to be uploaded as a single consolidated pdf containing all answers.

1 This question relates to the 3C5 laboratory experiment "Gyroscopic Phenomena". The frequency of nutation p for the gyro when inclined at angle θ is given by

$$p = \frac{C\omega}{A}\left[1 + \frac{J_1}{A}cot^2(\theta) + \frac{I_1}{A}cosec^2(\theta)\right]^{-\frac{1}{2}}$$

where the rotor is AAC aligned with axes $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as shown in Fig. 1. The moment of inertia of the gyro assembly about $\boldsymbol{k}$ is J_1 (the moments of inertia about $\boldsymbol{i}$ and $\boldsymbol{j}$ are included in A). The moment of inertia of the stand is I_1 about the vertical $\boldsymbol{K}$. The spin rate of the rotor is ω which is considered to be 'fast' and constant.

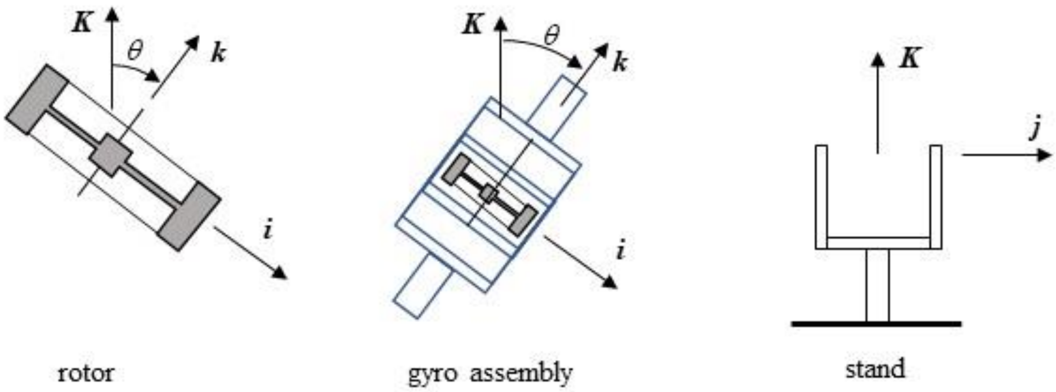

Fig. 1

(a) Sketch, for $0 < \theta < \pi$, the variation of p with θ for the case $I_1=A$ and $J_1=A$. Identify which of I_1 and J_1 is the more significant near $\theta = \pi/2$ and explain why this is the case. [15%]

(b) Identify the three components of the couple between the gyro assembly and the stand and explain why one of these is zero. [15%]

(c) With the aid of suitable diagrams and the Gyroscope Equations (for fast spin) write down equations relating the motion of the rotor, gyro assembly and stand. Use Euler angles θ, ϕ and ψ. [35%]

(d) Linearize these equations to find the frequency of small vibration around $\theta = \pi/2$ and show that your result is consistent with the expression for p given above. [35%]

2 A cylindrical solid body of radius a and height h is wobbling on a flat horizontal surface as shown in Fig. 2.

(a) For what value of h is the body "AAA" at the centre of mass G? [10%]

(b) During the wobbling motion, G is assumed to be at rest and the angle θ between the body's axis of symmetry and the vertical can be taken as constant and assumed to be small. The angular velocity of the body is described using the reference frame in Fig. 2 as $\underline{\omega} = \omega_1\underline{\mathrm{i}} + \omega_2\underline{\mathrm{j}} + \omega_3\underline{\mathrm{k}}$.

(i) Use a suitable no-slip condition to show that

$$\omega_3 = \frac{h}{2a}\dot{\phi}\sin\theta \qquad [30\%]$$

where $\dot{\phi}$ is the rate of rotation of the reference frame about the vertical axis $\underline{K}$.

(ii) For the case of steady-state wobbling, use a clear diagram to show that the couple acting on the body is

$$mg\left(\frac{h}{2}\sin\theta - a\cos\theta\right) \qquad [20\%]$$

(iii) Use the second Gyroscope equation (or otherwise) to find an expression for the rate $\dot{\phi}$ of steady-state wobbling and, using the result of part (a), find its value for the case of an "AAA" cylinder. [40%]

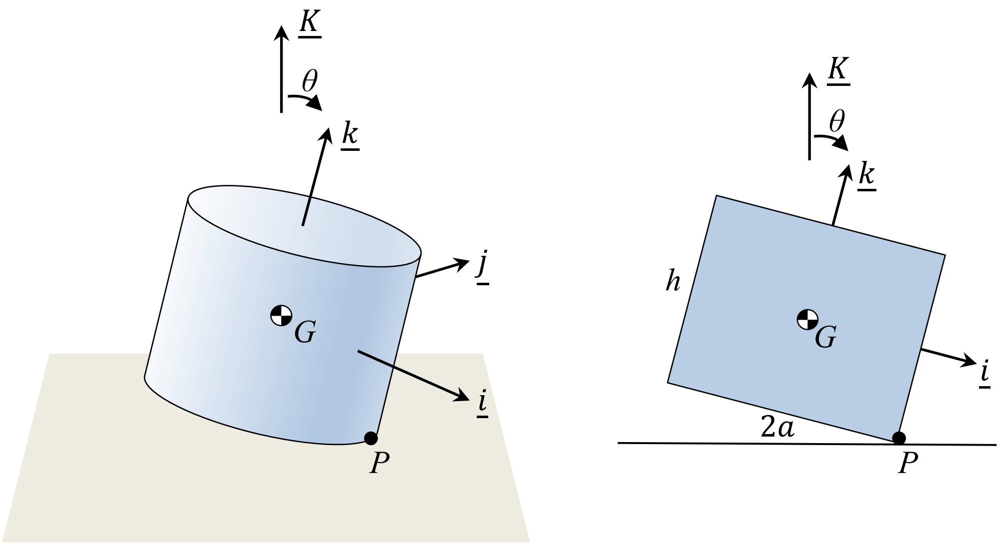

Fig. 2

3 A rigid circular ring of radius a is attached at O to a vertical shaft and shown in Fig. 3. The centre of the ring is at A and the fixed angle between OA and the shaft is β. The vertical shaft, and with it the ring, is forced to rotate about the vertical axis at constant angular velocity Ω. A small bead at P of mass m slides on the ring and its motion is described by ,θ the angle OAP. The acceleration due to gravity is g.

(a) Show that the kinetic energy of the bead can be expressed as

$$\mathrm{T} = \frac{1}{2}ma^2\left[\Omega^2\sin^2\theta + \left(\dot{\theta} + \Omega\sin\beta(1-\cos\theta)\right)^2\right]$$

and find and expression for the potential energy of the bead. [30%]

(b) Use Lagrange's Equation to find an equation of motion for the bead. Hence show that equilibrium solutions satisfy

$$[\mathrm{g}\cos\beta - a\Omega^2(\cos\theta\cos^2\beta + \sin^2\beta)]\sin\theta = 0$$ [20%]

(c) Find all possible equilibrium positions and identify the stability of two of them. Describe how their stability depends on the rotation speed Ω. [40%]

(d) Describe how the two regimes $\Omega^2 < \mathrm{g}\cos\beta/a$ and $\Omega^2 > \mathrm{g}\cos\beta/a$ differ. [10%]

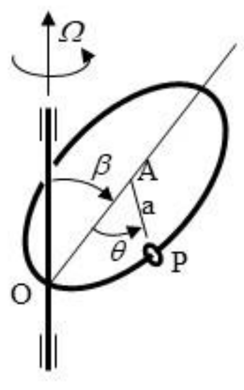

Fig. 3

4 (a) A wire of mass $6m$ and of length $12a$ is bent into the shape defined by points A to G, as shown in Fig. 4(a). Relative to Cartesian axes Oxyz, the coordinates of the points are as follows: A is $(-a, a, 2a)$; B is $(a, a, 2a)$; C is $(a, a, 0)$; D is $(a, -a, 0)$; E is $(-a, -a, 0)$; F is $(-a, -a, -2a)$ and G is $(-a, a, -2a)$. Find the moments of inertia I_{zz} and I_{xy}. [50%]

(b) Two trolleys of mass M_1 and M_2 are connected by a spring of stiffness K, as shown in Fig. 4(b). The motion of the system is described by two degrees of freedom, consisting of the displacement of the left hand trolley q_1 and the stretch of the spring q_2.

(i) Find expressions for the kinetic and potential energies of the system in terms of the specified degrees of freedom. [15%]

(ii) Find expressions for the generalised momenta, and hence show that the Hamiltonian of the system is given by

$$H = \frac{1}{2}\begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} M_1 + M_2 & M_2 \\ M_2 & M_2 \end{bmatrix}^{-1} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \frac{1}{2} K q_2^2$$ [20%]

(iii) Derive Hamilton's equations of motion, and explain why one of the generalised momenta is conserved during the motion of the system. [15%]

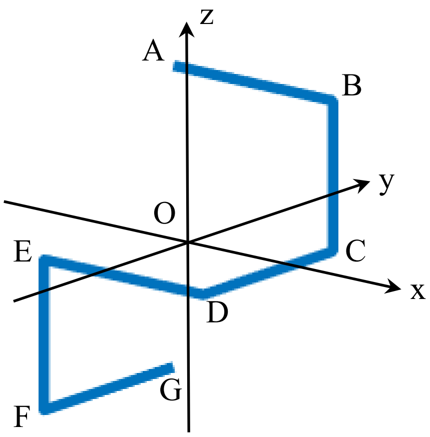

Fig. 4(a)

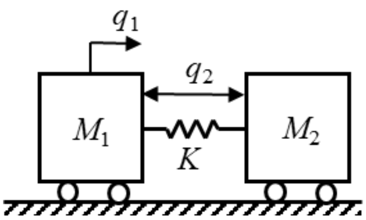

Fig. 4(b)

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Part IIA Data Sheet

Module 3C5 Dynamics
Module 3C6 Vibration

1 Dynamics in three dimensions

1.1 Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\boldsymbol{v}_{\mathrm{G}}$, M is the total mass, $\boldsymbol{v}_{\mathrm{G}}$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_{\mathrm{G}}$$

$$= \dot{\boldsymbol{h}}_{\mathrm{P}} + \dot{\boldsymbol{r}}_{\mathrm{P}} \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here $\boldsymbol{h}_{\mathrm{P}}$ and $\boldsymbol{h}_{\mathrm{G}}$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{P}}) \times m_i \dot{\boldsymbol{r}}_i$$

$$= \boldsymbol{h}_{\mathrm{G}} + (\boldsymbol{r}_{\mathrm{G}} - \boldsymbol{r}_{\mathrm{P}}) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}) dm = \boldsymbol{I\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz \; dm \qquad E = \int zx \; dm \qquad F = \int xy \; dm$

where all integrals are taken over the volume of the body.

1.2 Axes rotating with angular velocity $\boldsymbol{\Omega}$

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1.1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

1.3 Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= Q_1 \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= Q_2 \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= Q_3 \end{aligned}$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the"Gyroscope equations"):

$$\begin{aligned} A\dot{\Omega}_1 - (A\Omega_3 - C\omega_3)\Omega_2 &= Q_1 \\ A\dot{\Omega}_2 + (A\Omega_3 - C\omega_3)\Omega_1 &= Q_2 \\ C\dot{\omega}_3 &= Q_3 \end{aligned}$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\boldsymbol{\Omega} = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

1.4 Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy and Q_i are the non-conservative generalised forces.

1.5 Hamilton's equations

(a) Basic formulation

The generalized momenta p_i and the Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$ are defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}, \qquad H(\boldsymbol{p}, \boldsymbol{q}) = \sum_i p_i \dot{q}_i - T + V$$

where it should be noted that in the expression for the Hamiltonian the velocities $\dot{q}_i(\boldsymbol{p}, \boldsymbol{q})$ must be expressed as a function of the generalized momenta and the generalized displacements.

Hamilton's equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i.$$

(b) Extension topics

The total time derivative of some function $f(\boldsymbol{p}, \boldsymbol{q}, t)$ can be expressed in terms of the Poisson bracket $\{f, H\}$ in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\}, \qquad \{f, H\} \equiv \sum_i \left[\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right].$$

Common forms of Canonical Transform for Hamilton's equations are:

Type	Generating function	1st eqn	2nd eqn	Kamiltonian
1	$G_1(\boldsymbol{q}, \boldsymbol{Q}, t)$	$\boldsymbol{p} = \frac{\partial G_1}{\partial \boldsymbol{q}}$	$\boldsymbol{P} = -\frac{\partial G_1}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_1}{\partial t}$
2	$G_2(\boldsymbol{q}, \boldsymbol{P}, t)$	$\boldsymbol{p} = \frac{\partial G_2}{\partial \boldsymbol{q}}$	$\boldsymbol{Q} = \frac{\partial G_2}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_2}{\partial t}$
3	$G_3(\boldsymbol{p}, \boldsymbol{Q}, t)$	$\boldsymbol{q} = -\frac{\partial G_3}{\partial \boldsymbol{p}}$	$\boldsymbol{P} = -\frac{\partial G_3}{\partial \boldsymbol{Q}}$	$K = H + \frac{\partial G_3}{\partial t}$
4	$G_4(\boldsymbol{p}, \boldsymbol{P}, t)$	$\boldsymbol{q} = -\frac{\partial G_4}{\partial \boldsymbol{p}}$	$\boldsymbol{Q} = \frac{\partial G_4}{\partial \boldsymbol{P}}$	$K = H + \frac{\partial G_4}{\partial t}$

2 Vibration modes and response

Discrete Systems / **Continuous Systems**

1. Equation of motion

Discrete Systems: The forced vibration of an N-degree-of-freedom system with mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ (both symmetric and positive definite) is governed by:

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}$$

where $\mathbf{y}$ is the vector of generalised displacements and $\mathbf{f}$ is the vector of generalised forces.

Continuous Systems: The forced vibration of a continuous system is determined by solving a partial differential equation: see Section 3 for examples.

2. Kinetic Energy

Discrete Systems:

$$T = \frac{1}{2}\dot{\mathbf{y}}^T\mathbf{M}\dot{\mathbf{y}}$$

Continuous Systems:

$$T = \frac{1}{2}\int \dot{y}^2 \mathrm{d}m$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Potential Energy

Discrete Systems:

$$V = \frac{1}{2}\mathbf{y}^T\mathbf{K}\mathbf{y}$$

Continuous Systems: See Section 3 for examples.

4. Natural frequencies and mode shapes

Discrete Systems: The natural frequencies ω_n and corresponding mode shape vectors $\mathbf{u}^{(n)}$ satisfy

$$\mathbf{K}\mathbf{u}^{(n)} = \omega_n^2\mathbf{M}\mathbf{u}^{(n)}$$

Continuous Systems: The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see Section 3) and boundary conditions, assuming harmonic time dependence.

5. Orthogonality and normalisation

Discrete Systems:

$$\mathbf{u}^{(j)T}\mathbf{M}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

$$\mathbf{u}^{(j)T}\mathbf{K}\mathbf{u}^{(k)} = \begin{cases} 0 & j \neq k \\ \omega_j^2 & j = k \end{cases}$$

Continuous Systems:

$$\int u_j(x)u_k(x)\mathrm{d}m = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

6. General response

Discrete Systems: The general response of the system can be written as a sum of modal responses:

$$\mathbf{y}(t) = \sum_{j=1}^{N} q_j(t)\mathbf{u}^{(j)} = \mathbf{U}\mathbf{q}(t)$$

where $\mathbf{U}$ is a matrix whose N columns are the normalised eigenvectors $\mathbf{u}^{(j)}$ and q_j can be thought of as the 'quantity' of the jth mode.

Continuous Systems: The general response of the system can be written as a sum of modal responses:

$$y(x,t) = \sum_j q_j(t)u_j(x)$$

where $y(x,t)$ is the displacement and q_j can be thought of as the 'quantity' of the jth mode.

7. Modal coordinates

Modal coordinates $\mathbf{q}$ satisfy:

$$\ddot{\mathbf{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\mathbf{q} = \mathbf{Q}$$

where $\mathbf{y} = \mathbf{U}\mathbf{q}$ and the modal force vector $\mathbf{Q} = \mathbf{U}^T\mathbf{f}$.

Each modal amplitude $q_j(t)$ satisfies:

$$\ddot{q}_j + \omega_j^2 q_j = Q_j$$

where $Q_j = \int f(x,t)u_j(x)\mathrm{d}m$ and $f(x,t)$ is the external applied force distribution.

8. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k, the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x_1, and displacement y measured at point x_2, the transfer function is

$$H(x_1,x_2,\omega) = \frac{y}{F} = \sum_n \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x_1,x_2,\omega) = \frac{y}{F} \approx \sum_n \frac{u_n(x_1)u_n(x_2)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping), where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

9. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with well-separated resonances (low modal overlap), if the factor $u_n(x_1)u_n(x_2)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no anti-resonance.

10. Impulse responses

For a unit impulsive generalised force $f_j = \delta(t)$, the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x_1, the response at point x_2 is

$$g(x_1,x_2,t) = \sum_n \frac{u_n(x_1)u_n(x_2)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x_1,x_2,t) \approx \sum_n \frac{u_n(x_1)u_n(x_2)}{\omega_n} e^{-\omega_n\zeta_n t} \sin\omega_n t$$

for $t \geq 0$ (with small damping).

11. Step response

For a unit step generalised force f_j applied at $t = 0$, the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - \cos \omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\Big]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x_1, the response at point x_2 is

$$h(x_1, x_2, t) = \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - \cos \omega_n t\Big]$$

for $t \geq 0$ (with no damping), or

$$h(x_1, x_2, t) \approx \sum_n \frac{u_n(x_1) u_n(x_2)}{\omega_n^2} \Big[1 - e^{-\omega_n \zeta_n t} \cos \omega_n t\Big]$$

for $t \geq 0$ (with small damping).

2.1 Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is

$$\frac{V}{\tilde{T}} = \frac{\mathbf{y}^T \mathbf{K} \mathbf{y}}{\mathbf{y}^T \mathbf{M} \mathbf{y}}$$

where $\mathbf{y}$ is the vector of generalised coordinates (and $\mathbf{y}^T$ is its transpose), $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions in Section 3.

If this quantity is evaluated with any vector $\mathbf{y}$, the result will be
(1) $\geq$ the smallest squared natural frequency;
(2) $\leq$ the largest squared natural frequency;
(3) a good approximation to ω_k^2 if $\mathbf{y}$ is an approximation to $\mathbf{u}^{(k)}$.

Formally $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.

3 Governing equations for continuous systems

3.1 Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $y(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 y}{\partial t^2} - P\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.2 Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $\tau(x,t)$ per unit length. The polar moment of area is given by $J = (\pi/2)\big(a^4 - b^4\big)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = \tau(x,t)$	$V = \dfrac{1}{2}GJ\displaystyle\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\displaystyle\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

3.3 Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $y(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} - EA\dfrac{\partial^2 y}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\displaystyle\int\left(\dfrac{\partial y}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

3.4 Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $y(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 y}{\partial t^2} + EI\dfrac{\partial^4 y}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\displaystyle\int\left(\dfrac{\partial^2 y}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\displaystyle\int\left(\dfrac{\partial y}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

The first non-zero solutions for the following equations have been obtained numerically and are provided as follows:

$$\begin{aligned}
\cos\alpha\cosh\alpha + 1 &= 0, & \alpha_1 &= 1.8751\\
\cos\alpha\cosh\alpha - 1 &= 0, & \alpha_1 &= 4.7300\\
\tan\alpha - \tanh\alpha &= 0, & \alpha_1 &= 3.9266
\end{aligned}$$