EGT2
ENGINEERING TRIPOS PART IIA

Thursday 28 April 2022 2 to 3.40

Module 3C5

DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Attachment: 3C5 Dynamics and 3C6 Vibration datasheet 2021 (7 pages)
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

You may not remove any stationery from the Examination Room.

1 A spinning top of mass m is shown in Fig. 1. Its centre of mass G lies a distance a from the fixed point O which is in contact with a table. The top has principal moments of inertia A, A, C about O. The position of the top is described by the usual Euler angles θ, ϕ, ψ as shown.

(a) In terms of the Euler angles write down expressions for the angular velocities $\Omega_1, \Omega_2, \Omega_3$ of the reference frame and $\omega_1, \omega_2, \omega_3$ of the body. Explain why $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$ and write down the value of $\omega_3 - \Omega_3$. [20%]

(b) Use the Gyroscope equations from the data sheet to determine for fast spin the rate of steady precession of the top. [20%]

(c) If the precessing top is subject to a small perturbation determine the frequency of small oscillations, again for fast spin. [30%]

(d) What is the minimum spin speed ω_3 required for the top to stay upright? [30%]

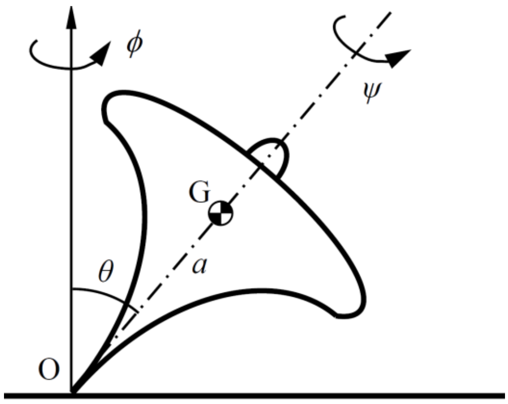

Fig. 1

2 A thin disc of mass m has principal moments of inertia A, A, C about its centre of mass. It is moving freely through space under the influence of no external couples. The components of the disc's angular velocity are denoted ω_1, ω_2 and ω_3 in a body-fixed reference frame aligned with the principal axes.

(a) Use Euler's equations

(i) to show that ω_3 is constant and [10%]

(ii) to obtain a pair of coupled differential equations involving ω_1 and ω_2. [30%]

(b) Show that the motion describing ω_1 and ω_2 is oscillatory (SHM) and show that the wobbling rate of the disc is equal to ω_3. [20%]

(c) Use the Gyroscope equations to obtain the frequency of small-amplitude nutation and explain carefully why this is not the same as your answer in (b). [40%]

3 (a) A Rubik’s Cube of mass $27m$ and side $3a$ *is* shown in Fig. 2. It is made up of 27 small uniform solid cubes each of mass m and side a .

(i) Find the principal moments of inertia of the Cube at the vertex P shown in Fig. 2(a). [25%]

(ii) The “top face” of the puzzle (comprising 9 cubes) is rotated through an angle of 45° as shown in Fig 2(b). Find the principal moments of inertia at point Q of the face-rotated Cube. [25%]

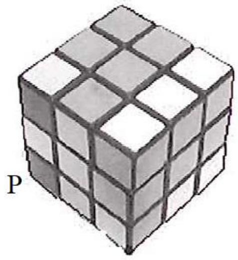

Fig. 2(a)

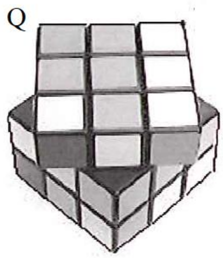

Fig. 2(b)

(b) A freely flying particle of mass M has a horizontal position $x(t)$ relative to a reference point, and a vertical position $y(t)$ above the ground.

(i) Find expressions for the generalized momenta associated with x and y. [5%]

(ii) Derive the Hamiltonian for the motion. [15%]

(iii) Demonstrate that the quantity $\dot{x}^2(\dot{y}^2 + 2gy)$ is conserved during the motion, using the definition of Poisson brackets given on the Data Sheet. [20%]

(iv) In physical terms, why is this a conserved quantity? [10%]

4 A hollow tube is pivoted about one end so that it can rotate in a vertical plane, as shown in Fig. 3. A mass on a spring is placed inside the tube, with the spring being connected to the pivot point. The spring is of stiffness K and unstretched length a, and the sprung mass is M. The hollow tube is of length L and has mass per unit length ρ. The extension of the spring is $x(t)$ and angle of the tube to the vertical is $\theta(t)$.

(a) Find expressions for the kinetic and potential energies of the system, and also for the generalised momenta associated with $x(t)$ and $\theta(t)$. Give a physical interpretation for each momentum. [25%]

(b) By using Lagrange's equation, derive the equations of motion of the system and verify that the equation governing $\theta(t)$ has the form

$$(\rho/3)L^3\ddot{\theta}+M(a+x)^2\ddot{\theta}+2M(a+x)\dot{x}\dot{\theta}+Mg(x+a)\sin\theta+(\rho g/2)L^2\sin\theta=0.$$ [35%]

(c) Your equations can be checked by looking at special cases. Simplify the equations for each of the following cases:

(i) $M=x=0$;
(ii) $\rho=x=0$;
(iii) $\theta=0$;
(iv) $g=a=K=\rho=0$

and for each case identify with a sketch the system described. For (i) to (iii) write down the natural frequency of small vibration. [40%]

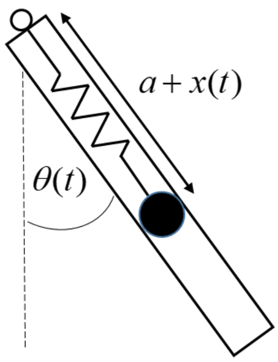

Fig. 3

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