Question 1

1 A shaft of length L, cross-sectional area A and polar moment of area J is made of a material with density ρ and shear modulus G, as illustrated in Fig. 1. One end of the shaft is prevented from rotating (at $x = 0$) and the other end is free (at $x = L$). The shaft can undergo small amplitude torsional oscillations.

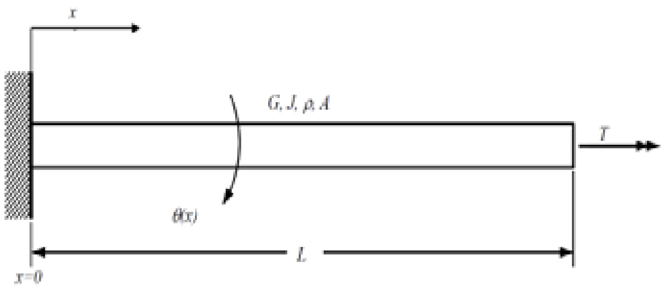


Fig. 1

(a) Write down the appropriate boundary conditions at each end and find expressions for the mode shapes $u_n(x)$ and natural frequencies ω_n. [20%]

BC's: $x = 0, \quad \theta(0,t) = 0$ —— ①

$x = L, \quad GJ\theta'(L,t) = 0.$ —— ②

mode shapes: wave equation solutions sinusoidal

hence $u_n(x) = A_n \cos k_n x + B_n \sin k_n x.$

① ⇒ $A_n = 0.$

② ⇒ $u_n'(x) = B_n k_n \cos k_n x$

$B_n k_n \cos k_n L = 0$

$\cos k_n L = 0.$

$k_n L = (n - \tfrac{1}{2})\pi$

Giving: $u_n(x) = B_n \sin \dfrac{(n-\frac{1}{2})\pi x}{L}$ //.

for ω_n need relationship with k_n.

wave equation: $\omega_n = c_p k_n = \sqrt{G/\rho}\, k_n.$

c_p — phase velocity.

hence $\omega_n = \dfrac{(n-\frac{1}{2})\pi}{L}\sqrt{G/\rho}$ //

(b) A torque T is applied at the free end, $x = L$.

(i) Derive an expression for the transfer function $H_1(L, x, \omega)$ from T to the output angular displacement θ at an arbitrary position x. Your answer should be expressed as a summation and should be in terms of the properties of the shaft. [20%]

$$H_1(L,x,\omega) = \sum_n \frac{u_n(x)\, u_n(L)}{\omega_n^2 - \omega^2}$$

mass normalise mode shape: $u_n = B_n \sin k_n x$.

$$\int u_n^2(x)\, dm = 1 \quad , \quad dm = \underbrace{\rho J}_{\text{inertia per unit length}} dx$$

$$\int_0^L B_n^2 \sin^2 k_n x \,.\, \rho J dx = 1$$

$$\frac{\rho J B_n^2}{2} \int_0^L (1 - \cos 2k_n x)\, dx$$

$$\frac{\rho J B_n^2}{2} L = 1 \quad \Rightarrow \quad B_n = \sqrt{\frac{2}{\rho J L}}$$

$$u_n(x) = \sqrt{\frac{2}{\rho J L}} \sin k_n x \,,$$

$$u_n(L) = \sqrt{\frac{2}{\rho J L}} \sin k_n L = \sqrt{\frac{2}{\rho J L}} \sin(n - \tfrac{1}{2})\pi = (-1)^{n+1} \sqrt{\frac{2}{\rho J L}}$$

so $$H_1(L,x,\omega) = \frac{2}{\rho J L} \sum_{\text{all } n} \frac{\sin k_n x \,.\, \sin k_n L}{\omega_n^2 - \omega^2} \quad \text{OR} \quad \frac{2}{\rho J L} \sum_{\text{all } n} \frac{(-1)^{n+1} \sin k_n x}{\omega_n^2 - \omega^2}$$

$$\left(\begin{array}{l} \text{where } k_n = (n - \tfrac{1}{2})\pi/L \\ \& \ \omega_n = \dfrac{(n - \tfrac{1}{2})\pi}{L} \sqrt{G/\rho} \end{array} \right)$$

Question 1 (continued)

(ii) By appropriately differentiating your answer to (b)(i) or otherwise, derive an expression for the transfer function $H_2(L, 0, \omega)$ from T to the reaction torque at the boundary $x = 0$. [20%]

$$H_1(L, x, \omega) = \frac{\theta(x)}{T(L)}$$

now $T = GJ\theta'$,

so
$$\frac{T(x)}{T(L)} = \frac{GJ\theta'(x)}{T(L)}$$
$$= GJ\frac{\partial H_1(L,x,\omega)}{\partial x}$$

$$u_n(x) = \sqrt{\frac{2}{\rho J L}} \sin k_n x$$
$$u_n'(x) = \sqrt{\frac{2}{\rho J L}} k_n \cos k_n x.$$

$u_n'(0)$ $\Uparrow$ $\quad$ $\sin k_n L$ $\Uparrow$

so
$$H_2(L,0,\omega) = \frac{2}{\rho \cancel{J} L} \cdot G\cancel{J} \sum_n \frac{k_n \cdot (-1)^{n+1}}{\omega_n^2 - \omega^2}$$

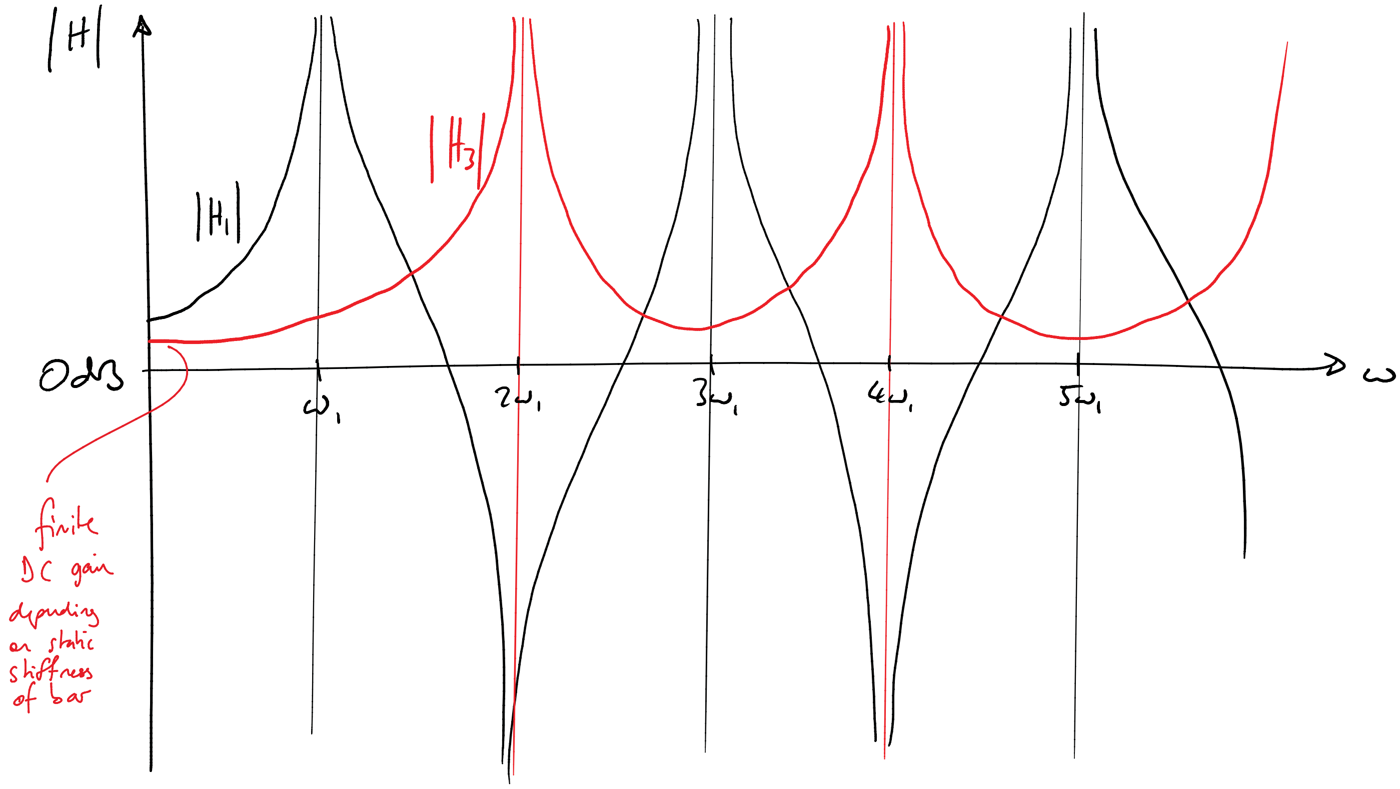
(iii) Find the transfer function $H_3(L, 0, \omega)$ from input angular displacement at $x = L$ to reaction torque at $x = 0$ in terms of $H_1(L, L, \omega)$ and $H_2(L, 0, \omega)$, noting that H_1 is evaluated at the output position $x = L$. [10%]

$$H_1(L,L,\omega) = \frac{\theta(L)}{T(L)}, \qquad H_2(L,0,\omega) = \frac{T(0)}{T(L)}$$

so
$$H_3(L,0,\omega) = \frac{T(0)}{\theta(L)} = \frac{T(0)}{T(L)}\frac{T(L)}{\theta(L)}$$
$$= \frac{H_2(L,0,\omega)}{H_1(L,L,\omega)}$$

Question 1 (continued)

(iv) Sketch the transfer functions $H_1(L, L, \omega)$ and $H_3(L, 0, \omega)$, labelling both resonant and anti-resonant frequencies. [30%]



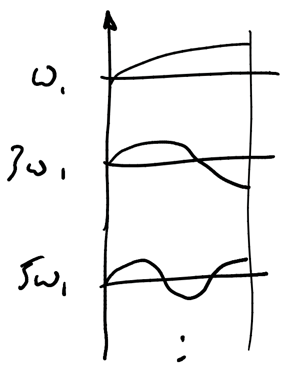
Notes: H_1 resonances @ ω_1, $3\omega_1$, $5\omega_1$. & Driving point response so antiresonance between each peak.

H_2 resonances also @ ω_1, $3\omega_1$, $5\omega_1$ (from TF expression in (b)(ii)). No antiresonance as sign of numerator swaps for each peak.

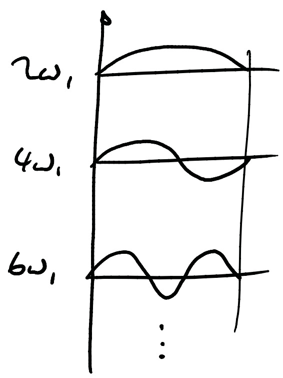
$H_3 = H_2/H_1$. So peaks at antiresonance of H_1. Peaks in H_2 & H_1 will cancel. This can be shown formally by approximating the series by the single term near a given peak.

Alternative perspective: H_3 resonances as per fixed-fixed torsional vibration, so first mode @ $2\omega_1$ & regular spacing.

H_1 mode shapes



H_3 mode shapes



Analytic derivation (additional note)

$$\theta(x,t) = U(x)\, e^{i\omega t}$$

BC's $\Rightarrow$ $\theta(0,t) = 0$, $\theta(L,t) = \theta_L e^{i\omega t}$ ← input displacement.

$$U(x) = A \sin kx + B \cos kx.$$

$$B = 0$$

$$A \sin kL = \theta_L \Rightarrow A = \frac{\theta_L}{\sin kL}.$$

hence $U(x) = \dfrac{\theta_L}{\sin kL} \cdot \sin kx.$

want torque output at $x = 0$, $T = GJ\theta'$

so $T(0) = GJ\theta'(0) = \dfrac{GJ\theta_L}{\sin kL} \cdot k \cos k0$

$$= \frac{GJk\theta_L}{\sin kL}$$

hence $H_3 = \dfrac{T(0)}{\theta_L(L)} = \dfrac{GJk}{\sin kL} = \dfrac{GJ\,\omega/c}{\sin \omega L/c} \propto \dfrac{1}{\sin \omega L/c}$

which will give same sketch above

Question 2

2 A beam of length L and uniform cross-section with second moment of area I, is made of material with density ρ and Young's modulus E. The beam is clamped at both ends and undergoes small-amplitude transverse vibration.

(a) Starting from the governing equation for transverse vibration of a beam, derive an expression whose solutions give the wavenumbers k_n for the modes of the beam. [20%]

from datasheet: $\rho A \ddot{y} - EI y^{iv} = 0$

let $y = U(x) e^{i\omega t}$

$\rightarrow$ PDE $\Rightarrow$ $+\rho A \omega^2 U + EI U^{iv} = 0.$

$$U^{iv} + \underbrace{\omega^2 \rho A / EI}_{= k^4} U = 0$$

solution: $U = D_1 \sin kx + D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx$

$U' = k\left(D_1 \cos kx - D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx\right)$

$U'' = k^2\left(-D_1 \sin kx - D_2 \cos kx + D_3 \sinh kx + D_4 \cosh kx\right)$

$U''' = k^3\left(-D_1 \cos kx + D_2 \sin kx + D_3 \cosh kx + D_4 \sinh kx\right)$

BC's: @ $x = 0, L$, $U = U' = 0.$

$x = 0$, $U = 0 \Rightarrow D_2 + D_4 = 0 \Rightarrow D_2 = -D_4$

$U' = 0 \Rightarrow D_1 + D_3 = 0 \Rightarrow D_1 = -D_3$

$x = L$, $U = 0 \Rightarrow$, $U' = 0 \Rightarrow$
$$\begin{bmatrix} \sin kL - \sinh kL & \cos kL - \cosh kL \\ \cos kL - \cosh kL & -\sin kL - \sinh kL \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

natural modes when det $= 0$:

$$-\left(\sin kL + \sinh kL\right)\left(\sin kL - \sinh kL\right) - \left(\cos kL - \cosh kL\right)^2 = 0$$

Question 2 (continued)

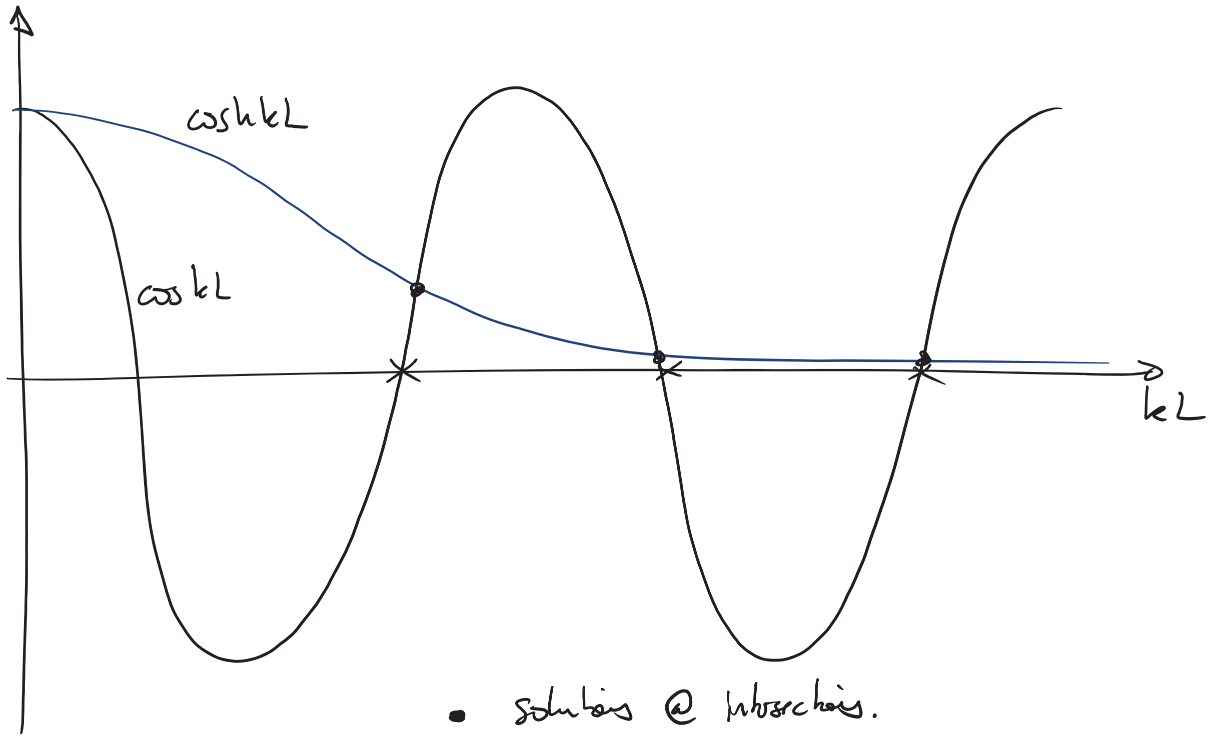
$$-(\sin kL + \sinh kL)(\sin kL - \sinh kL) - (\cos kL - \cosh kL)^2 = 0$$

$$\sinh^2 kL - \sin^2 kL - \cosh^2 kL - \cos^2 kL + 2\cos kL \cosh kL = 0 .$$

$$= -1 \qquad = -1$$

Giving $\cos kL \cosh kL = +1$ // solutions $k = k_n$

(b) Using a graphical construction, estimate the first six natural frequency ratios of the beam, i.e. estimate ω_n/ω_1 for $1 \le n \le 6$. [20%]



• solutions @ intersections.

× approx solutions.

approx: $k_n L = (n + \frac{1}{2})\pi$ (improves with increasing n).

Question 2 (continued)

$$k_n L \simeq \left(n+\tfrac{1}{2}\right)\pi = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \ldots$$

$$\& \quad k^4 = \omega^2 \left(\rho A / EI \right)$$

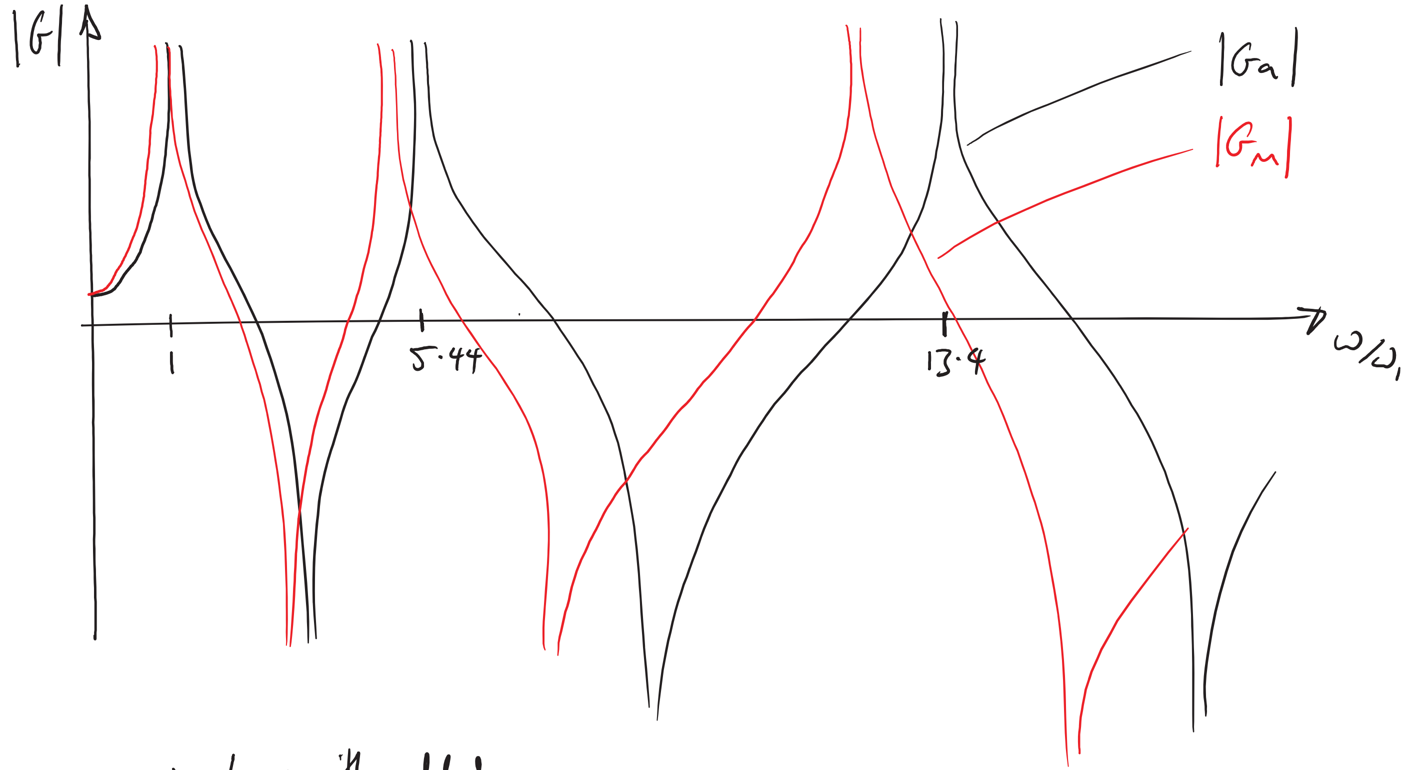
$$\text{so} \quad \omega_n = k_n^2 \sqrt{\frac{EI}{\rho A}}, \quad \text{ie} \quad \omega_n \propto k_n^2 .$$

$$\omega_n \propto 3^2, 5^2, 7^2, 9^2, 11^2, 13^2$$

$$\frac{\omega_n}{\omega_1} \simeq 1, 2.78, 5.44, 9, 13.4, 18.8$$

(c) A measurement is carried out to find the driving point transfer function at the centre of the beam G_a. An instrumented hammer is used to strike the beam at the centre, and the response is measured using an accelerometer mounted at the centre. The accelerometer has a small but non-zero mass m, resulting in a measured transfer function G_m.

(i) Sketch the driving point transfer functions G_m and G_a, i.e. both with and without including the effect of the added mass of the accelerometer. Include in your sketch frequencies up to and including ω_6. [20%]



- ω_n lower with added mass
- small change @ low freq, large change @ high frequency
- only symmetric modes visible.

(ii) By considering the coupled driving point transfer function at the accelerometer, derive an expression to calculate the actual transfer function G_a (without the mass) from the measured transfer function G_m (with the mass). [30%]

$$G_{coupled} = G_m = \left(\frac{1}{G_a} + \frac{1}{G_{mass}} \right)^{-1} , \quad G_{mass} = \frac{-1}{m\omega^2}$$

$$\text{so} \quad G_m = \left(\frac{1}{G_a} - m\omega^2 \right)^{-1}$$

$$= \left(\frac{1 - m\omega^2 G_a}{G_a} \right)^{-1} = \frac{G_a}{1 - m\omega^2 G_a} .$$

$$(1 - m\omega^2 G_a) G_m = G_a .$$

$$G_a (1 + m\omega^2 G_m) = G_m$$

$$\text{so} \quad G_a = \frac{G_m}{1 + m\omega^2 G_m} \quad //.$$

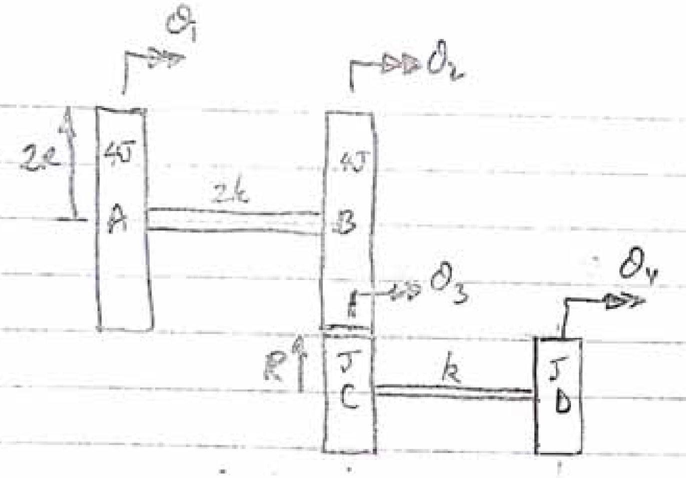
(iii) Under what conditions will the mass-compensated estimate be best and why? [10%]

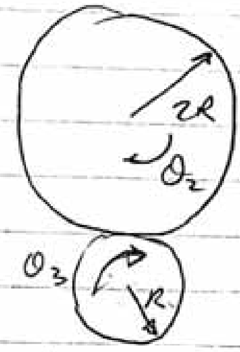
good estimate when mass has small effect, ie $G_a \simeq G_m$.

This happens for small mass, low frequency

ie $m\omega^2 \ll 1/G_m$ //.

3.





Gear ratio:

$$2R\,\theta_2 = -R\,\theta_3$$
$$\rightarrow \quad \theta_3 = -2\theta_2$$

Potential Energy:

$$V = \tfrac{1}{2}\left\{2k(\theta_2-\theta_1)^2 + \tfrac{1}{2}k(\theta_4-\theta_3)^2\right\}$$

$$= \tfrac{1}{2}k\left\{2\theta_1^2 + 2\theta_2^2 - 4\theta_1\theta_2 + \theta_4^2 + 4\theta_2^2 + 4\theta_2\theta_4\right\}$$

$$= \tfrac{1}{2}\begin{bmatrix}\theta_1 & \theta_2 & \theta_4\end{bmatrix} k \underbrace{\begin{bmatrix}2 & -2 & 0\\ -2 & 6 & +2\\ 0 & +2 & 1\end{bmatrix}}_{[K]} \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_4\end{Bmatrix}$$

Kinetic Energy:

$$2\dot{\theta}_2$$

$$T = \tfrac{1}{2}\left\{4J\dot{\theta}_1^2 + 4J\dot{\theta}_2^2 + J\dot{\theta}_3^2 + J\dot{\theta}_4^2\right\}$$

$$= \tfrac{1}{2}\begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_4\end{bmatrix} J \underbrace{\begin{bmatrix}4 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 1\end{bmatrix}}_{[M]} \begin{Bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_4\end{Bmatrix}$$

Eigenvalues:

$$\left([K] - \omega^2[M]\right)\underline{u} = 0$$

$$\begin{bmatrix}2k-\omega^2 4J & -2k & 0\\ -2k & 6k-\omega^2 8J & +2k\\ 0 & +2k & k-\omega^2 J\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_4\end{Bmatrix} = 0 \quad \text{——— (1)}$$

$\Rightarrow$ det $=0$ $\quad [2k-\omega^2 4J]\left[(6k-\omega^2 8J)(k-\omega^2 J) - 4k^2\right] + 2k\left[(-2k)(k-\omega^2 J)\right] = 0$

3 cont $\quad (2k-\omega^2 4J)(2k^2 - 14kJ\omega^2 + 8\omega^4J^2) - 4k^2(k-\omega^2 J) = 0$

$$\cancel{4k^3} - 28k^2J\omega^2 + 16\omega^4 kJ^2 - 8k^2J\omega^2 + 56J^2k\omega^4 - 32\omega^6J^3$$
$$-\cancel{4k^3} + 4\omega^2k^2J \quad \Rightarrow \quad \underline{\underline{\omega^2=0}}$$

$$k^2J(-28-8+4) + \omega^2kJ^2(16+56) - 32\omega^4J^3$$

$$32J^3\omega^4 - 72kJ^2\omega^2 + 32k^2J = 0$$

$$\text{ie} \quad 4\omega^4 - 9\left(\frac{k}{J}\right)\omega^2 + 4\left(\frac{k}{J}\right)^2 = 0$$

$$\omega^2 = \frac{9\left(\frac{k}{J}\right) \pm \frac{k}{J}\sqrt{81-64}}{8} = \frac{k}{J}\left[\frac{9\pm\sqrt{17}}{8}\right]$$

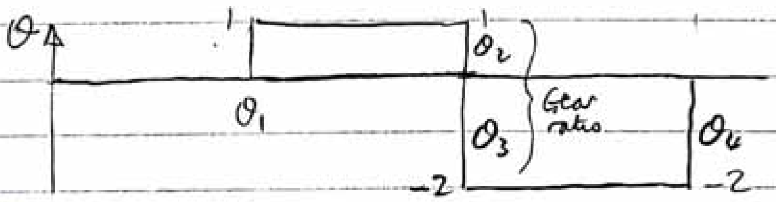
$$\text{ie} \quad \omega^2 = 0,\ 0.61\frac{k}{J},\ 1.64\frac{k}{J} \quad //$$

<u>Mode shapes</u>

<u>Mode 1</u> is a rigid body mode with $\omega_1^2 = 0$

The mode shape is $[\theta_1\ \theta_2\ \theta_4]^T = [1\ \ 1\ \ -2]$

Gear ratio



<u>Modes 2 & 3</u> : First row of ① $\quad (2k-\omega^2 4J)\theta_1 - 2k\theta_2 = 0$

Last row of ① $\quad 2k\,\theta_2 + (k-\omega^2 J)\theta_4 = 0$

$\omega_2^2 = 0.61\,k/J$:

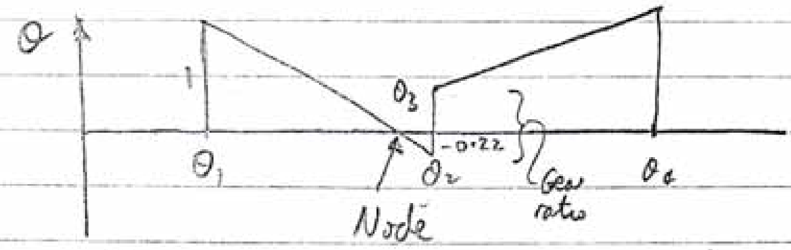
first row: $\quad (2k - 2.44k\omega^2)\theta_1 - 2k\theta_2 = 0 \Rightarrow \dfrac{\theta_2}{\theta_1} = -0.22$

Gear ratio $\quad \theta_3 = -2\theta_2 = 0.44\theta_1$

Last row: $\quad 2k\theta_2 + (k-0.61k)\theta_4 = 0 \Rightarrow \dfrac{\theta_4}{\theta_2} = -5.13$

3 Cont $$\frac{\theta_4}{\theta_1} = \frac{\theta_4}{\theta_2}\frac{\theta_2}{\theta_1} = (-5.13)(-0.22) = +1.13$$

So $[\theta_1\ \theta_2\ \theta_3\ \theta_4]^T = [1\ \ -0.22\ \ 0.44\ \ 1.13]$



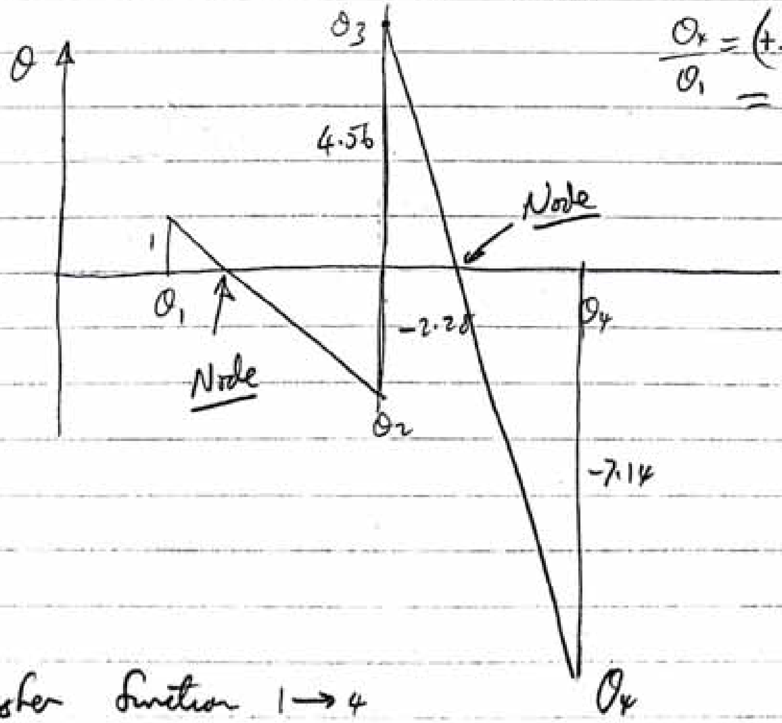
$\underline{\omega_3^2 = 1.64k/J}$

First row $$\frac{\theta_2}{\theta_1} = \frac{2k - 6.56k}{2k} = -2.28 \checkmark$$

Gear ratio $\theta_3 = -2\theta_2 = 4.56\,\theta_1$

Last row $$\frac{\theta_4}{\theta_2} = \frac{-2k}{k - 1.64k} = +3.13$$

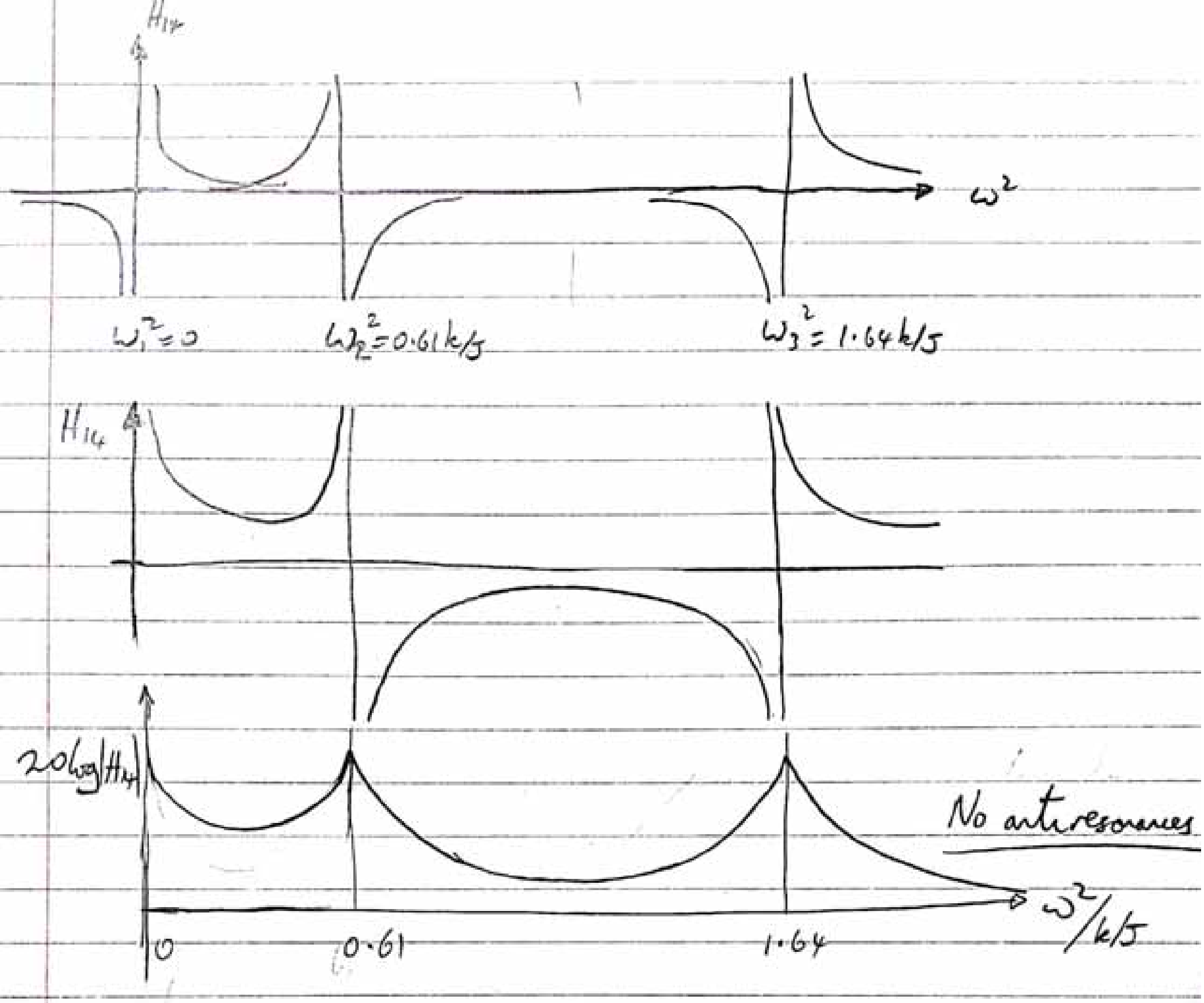
$$\frac{\theta_4}{\theta_1} = (+3.13)(-2.28) = -7.14$$



(c) Transfer function $1 \rightarrow 4$

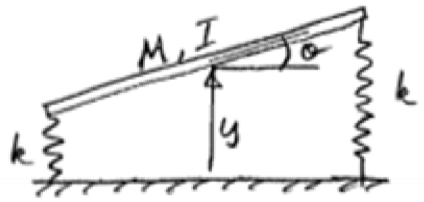
$$\frac{\theta_4}{T_1} = \sum_{n=1}^{3} \frac{u_1^{(n)} u_4^{(n)}}{\omega_n^2 - \omega^2}$$

Mode	Sign of $u_1^{(n)} u_4^{(n)}$
1	$-$
2	$+$
3	$-$



H_{14}
ω^2
$\omega_1^2 = 0$
$\omega_2^2 = 0.61k/J$
$\omega_3^2 = 1.64k/J$
H_{14}
$20\log|H_{14}|$
No antiresonances
0
0.61
1.64
$\omega^2/k/J$

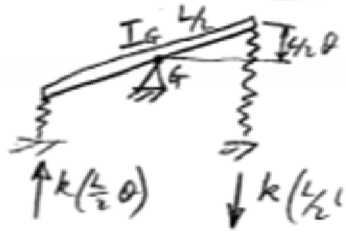
4. (a)



(i) Symmetric mode: $\theta = 0$ "bounce"

$$\underline{\underline{\omega_1^2 = 2k/m}}$$

(ii) Anti-symmetric mode: $y = 0$ "pitch"

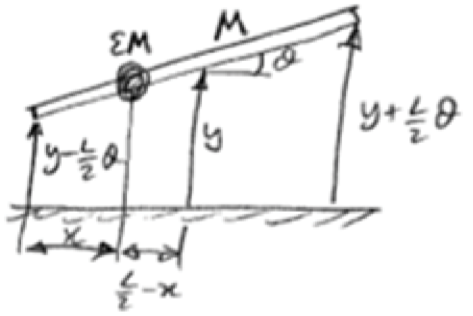


$$\Sigma M_G: \; 2\left(\tfrac{L}{2}\right) k \left(\tfrac{L}{2}\theta\right) = I_G \ddot{\theta}$$

with $I_G = \frac{1}{12} ML^2$

$$\Rightarrow \frac{k\cancel{L^2}}{\cancel{2}}\theta = \frac{M\cancel{L^2}}{\cancel{12}\,6}\ddot{\theta} \Rightarrow \underline{\underline{\omega_2^2 = \frac{6k}{M}}}$$

(b) With a small added mass ($\varepsilon \ll 1$) it is reasonable to assume that the mode shapes don't change significant from the modes in Part (a). ∴ Use these modes in Rayleigh's quotient.



PE $\quad V = \frac{1}{2}k\left[\left(y + \frac{L}{2}\theta\right)^2 + \left(y - \frac{L}{2}\theta\right)^2\right]$

KE $\quad T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I_G\dot{\theta}^2 + \frac{1}{2}(\varepsilon M)\left(\dot{y} - \left(\frac{L}{2} - x\right)\dot{\theta}\right)^2$

Mode (i) with $\theta = 0$:

$$\omega_1^2 \approx \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k\left[y^2 + y^2\right]}{\frac{1}{2}M\left[y^2 + \varepsilon(y^2)\right]} = \underline{\underline{\frac{k}{M}\frac{2}{(1+\varepsilon)}}}$$

Mode (ii) with $y = 0$:

$$\omega_2^2 \approx \frac{\frac{1}{2}k\left[\frac{L^2}{4}\theta^2 + \frac{L^2}{4}\theta^2\right]}{\frac{1}{2}\left(\frac{1}{12}ML^2\right)\left[\theta^2\right] + \frac{1}{2}\varepsilon M\left(\frac{L}{2} - x\right)^2\theta^2} = \frac{k}{M}\frac{L^2/2}{\frac{1}{12}L^2 + \varepsilon\left(\frac{L}{2} - x\right)^2}$$

$$= k/m \cdot \frac{1}{\frac{1}{6} + 2\varepsilon\left(\frac{1}{2} - x/L\right)^2} \approx \underbrace{\frac{6k}{m}}_{\text{old freq}} \underbrace{- \frac{2k}{m}\varepsilon\left(\frac{1}{2} - x/L\right)^2}_{\text{shift due to } \varepsilon M}$$

(using binomial expansion)

4(c) Shift in lower mode is independent of x, so the size of the difference in frequency between the two modes depends only on the value of ω_2^2.

(i) Largest difference is when ω_2^2 is highest, which happens when $x = L/2$ ie additional mass is in the middle. In this case ω_2^2 is unchanged from the value in part (a)

(ii) The smallest difference is when ω_2^2 is lowest, which happens when $x=0$ or $x=L$. At these positions the pitch moment of inertia of the system is largest.